Using linear algebra and graph theory for satisfiability decision

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What's it all about

We look at the SAT problem for clause-sets F.

We want to extract information from F which is useful for deciding satisfiability of F.

Thus, we must "forget" (ignore) much of F, and concentrate on a certain aspect.

In general, we must use many such "filters" ("microscopes", or "functors"). Here, we consider just one.

We concentrate on the aspect of the **conflict patterns**, where a conflict simply means a clashing (complementary) literal pair.

It turns out, that there is a rich algebraic-combinatorial structure which can be exploited.

We are also interested in transferring techniques from the area of satisfiability problems to algebra and combinatorics.

Hitting clause-sets

A **hitting clause-set** is a clause-set $F \in \mathcal{CLS}$ such that for all clauses $C_1, C_2 \in F$, $C_1 \neq C_2$ there is a literal x with $x \in C_1$ and $\overline{x} \in C_2$, i.e., $C_1 \cap \overline{C_2} \neq \emptyset$. For example

$$\big\{ \{a, b, c\}, \{\overline{a}, \overline{b}\}, \{a, \overline{c}\} \big\}$$

is a hitting clause-set.

A k-**uniform hitting clause-set** is a hitting clause-set F such that for all clauses $C_1, C_2 \in F$, $C_1 \neq C_2$ we have $|C_1 \cap \overline{C_2}| = k$, and a **uniform hitting clause-set** is a k-uniform hitting clause-set for some k. The above example is not a uniform hitting clause-set, while

$$\big\{ \{a, b, c\}, \{\overline{a}, b\}, \{a, \overline{c}\} \big\}$$

is a 1-uniform hitting clause-set.

The deficiency

Consider a clause-set $F \in \mathcal{CLS}$:

$$\begin{aligned} \boldsymbol{n(F)} &:= \text{ number of variables in } F \\ \boldsymbol{c(F)} &:= \text{ number of clauses in } F \\ \boldsymbol{\delta(F)} &:= c(F) - n(F) \text{ ``deficiency''} \\ \boldsymbol{\delta^*(F)} &:= \max_{F' \subseteq F} \delta(F') \text{ ``maximal deficiency''}. \end{aligned}$$

If F is minimally unsatisfiable, then $\delta(F) \geq 1$.

The maximal deficiency measures the complexity of F (in some sense):

1. the minimal number of resolution steps in a tree resolution refutation of unsatisfiable F is bounded by $n(F) \cdot 2^{\delta^*(F)-1}$;
2. if F is satisfiable, then there exists a partial assignment φ using at most $\delta^*(F)$ many variables, such that $\varphi * F$ is matching satisfiable (satisfiable by selecting a unique variable from each clause).

How it started

Hans Kleine Büning and Xishun Zhao in 2001 proved that a 1-uniform hitting clause-set F with $\delta(F) \geq 2$ must be satisfiable.

Their proof looked strange to me, and I guessed that simply for every 1-uniform hitting clause-set F we have $\delta(F) \leq 1$.

There was also the conjecture of Endre Boros from the 1998 SAT workshop, that every 1-uniform hitting clause-set F with $\delta(F) = 1$ must contain a variable occurring in one polarity exactly once.

Hans Kleine Büning and Xishun Zhao showed that from this conjecture in fact it follows that for every 1-uniform hitting clause-set F we have $\delta(F) \leq 1$.

(We will see, that Boros' conjecture is false, but the conclusion of Büning and Zhao is right.)

The addressing problem for graphs

Ronald Graham and H.O. Pollak in *On the addressing problem for loop switching* (1971) considered the problem of assigning addresses to nodes in a network, such that the

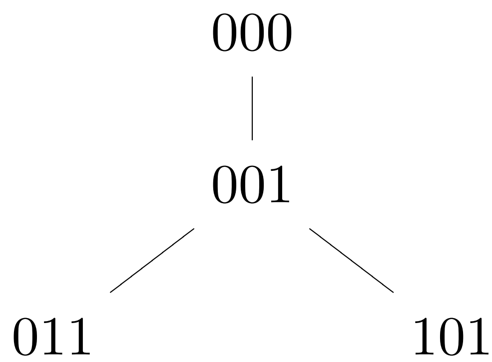
distances between two addresses
(measured in some appropriate way)
are equal to the
distances between the nodes in the network.

Packages can be sent through the network by comparing the destination address with the addresses of the neighbour switches in the network, and switch to a loop where the distance to the destination is decreased.

A "loop" is a two-way connection, and a "switch" is a node in the vertex, where one can switch from one loop to another loop. In graph theory, we just consider an undirected graph where

vertices are the switches
edges are the loops.

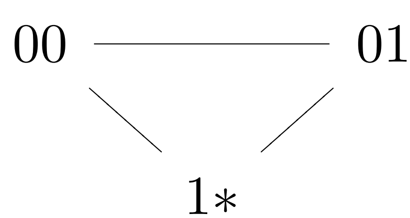
Using binary codes and Hamming distances for example:



Binary codes can only be used to address a connected graph if the graph is bipartite. The triangle can not be addressed using binary codes (and the Hamming distance).

In general, using a code with m symbols we can not address for example the K_{m+1}.

To have a universal addressing scheme, Graham and Pollak used a ternary code $\{0, 1, *\}$, and a modified Hamming distance d, which takes into account only positions without a star (the star means "ignore"):



The theorem of Graham and Pollak, and Witsenhausen's theorem

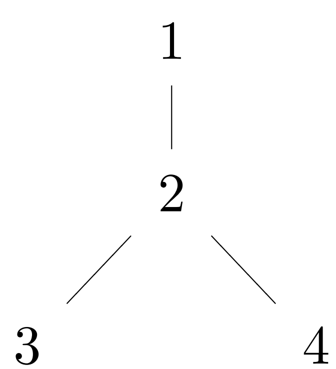
The addressing scheme uses a fixed addressing length.

It is natural to seek to minimise the number of positions.

Witsenhausen (apparently) showed that the K_m can not be addressed with less than $m-1$ positions.

More generally, Graham and Pollak considered the **distance matrix** (actually, they didn't, but I think it's better to do so) of a graph G, for example

$G =$



$$\mapsto D(G) = \begin{pmatrix} 0 & 1 & 2 & 2 \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & 2 \\ 2 & 1 & 2 & 0 \end{pmatrix}$$

and using Sylvester's Law of Inertia they proved, that any addressing of G needs at least

$$\max(i_-(D(G)), i_+(D(G)))$$

many positions, where for a symmetric real matrix A

$$\begin{aligned} i_-(A) &:= \text{number of negative eigenvalues of } A \\ i_+(A) &:= \text{number of positive eigenvalues of } A. \end{aligned}$$

The distance matrix of K_m is $J_m - I_m$, for example

$$D(K_3) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Easy linear algebra shows, that $J_m - I_m$ has the eigenvalue $m-1$ with multiplicity 1, and the eigenvalue -1 with multiplicity $m-1$, thus

$$i_-(D(K_m)) = m - 1,\ i_+(D(K_m)) = 1$$

and with the theorem of Graham and Pollak the theorem of Witsenhausen follows.

Back to clause-sets

Translating

- "0" as "positive"
- "1" as "negative"
- "$*$" as "not there"

every code over $\{0, 1, *\}$ using n positions can be translated into a clause-set over n variables, for example

$$\{000, 010, 1{*}{*}, 01{*}, {*}{*}{*}\} \mapsto \{ \{v_1, v_2, v_3\}, \{v_1, \overline{v_2}, v_3\}, \{\overline{v_1}\}, \{v_1, \overline{v_2}\}, \emptyset \}.$$

The modified Hamming distance becomes the number of clashing (or conflicting) literals.

Applying the theorem of Witsenhausen

The (symmetric) **conflict matrix** $\mathrm{C_s}(F)$ of a clause-set $F = \{C_1, \dots, C + m\}$ (with $c(F) = m$) is a symmetric matrix of order m with

$$\mathrm{C_s}(F)_{i,j} = |C_i \cap \overline{C_j}|.$$

For example

$$\mathrm{C_s}(\{\{a, b\}, \{\overline{a}, b\}, \{a, \overline{b}\}\}) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}.$$

A clause-set F is a 1-uniform hitting clause-set if and only if $\mathrm{C_s}(F) = J_{c(F)} - I_{c(F)}$ holds.

Every 1-uniform hitting clause-set F yields an addressing of $K_{c(F)}$, reverting the above translation, and thus by the theorem of Witsenhausen we get

$$\delta(F) \leq 1.$$

Counterexample to the Boros conjecture

We have just seen, that every 1-uniform hitting clause-set has deficiency at most one, as it was proven by Büning and Xishun under the assumption, that the conjecture of Boros is true. A counterexample to the conjecture of Boros now is given by the following clause-variable matrix

$$\begin{pmatrix} + & & + & & & & & + \\ + & & - & + & & & & \\ - & - & & & & + & + & \\ - & - & & & - & - & & \\ & + & - & - & - & & & \\ & & & - & + & & - & - \\ - & & & & + & - & & + \\ - & & & + & & + & - & \\ & + & + & & & & + & - \end{pmatrix}.$$

Every row represents a clause, every column a variable. There are nine rows and eight columns (i.e., nine clauses and eight variables), and every two rows have exactly one clash. Thus we have a 1-uniform hitting clause-set with deficiency one, and since every column contains exactly two $+$ and two $-$, there is no variable occurring exactly once in one polarity.

Generalising the approach

It seems, that the theorem of Graham and Pollak can be only applied to clause-sets if their conflict matrix is also the distance matrix of some graph.

In fact, many investigations originating from the work of Graham and Pollak have a strong geometric interest.

But the relation between codes over $\{0, 1, *\}$ of length n and clause-sets over variables, say, $v_1, \ldots, v_n$, is bijective, and the theorem of Graham and Pollak has nothing to do with any structure on the codes (or the clause-sets).

We define in general for any symmetric real matrix A the hermitian rank (Shader et al, 1999)

$$h(A) := \max(i_-(A), i_+(A)),$$

and for any clause-set F we have

$$n(F) \geq h(F) := h(\mathrm{C_s}(F)).$$

The conflict multigraph, and biclique decompositions

For a clause-set F we have defined the (symmetric) conflict matrix $\mathrm{C_s}(F)$.

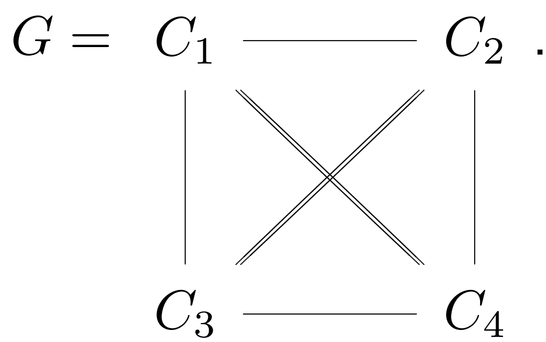
It is useful to consider this matrix as the adjacency matrix of the **conflict multigraph** of F:

- the vertices are the clauses of F;
- there are as many (parallel) edges between two clauses as there are conflicts between the clauses.

For example the conflict multigraph of $\{C_1, C_2, C_3, C_4\}$ for $C_1 = \{a, b\}$, $C_2 = \{\overline{a}, b\}$, $C_3 = \{a, \overline{b}\}$, $C_4 = \{\overline{a}, \overline{b}\}$ has symmetric conflict matrix

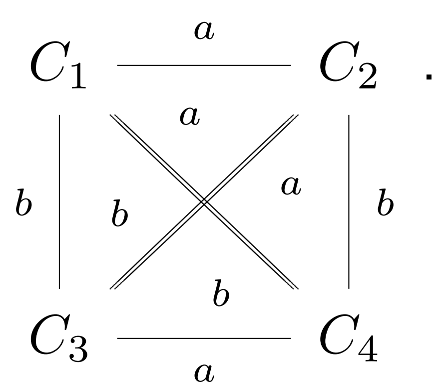
$$\begin{pmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{pmatrix}$$

while the conflict multigraph is

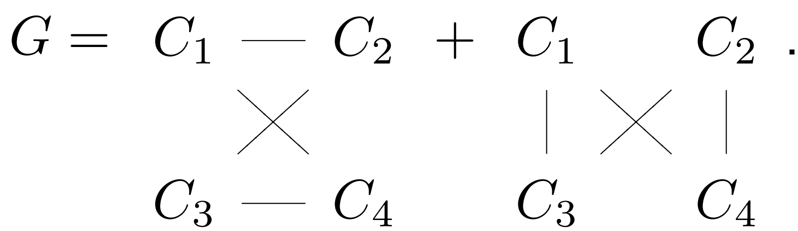


(note the parallel edges in the diagonals).

Now it is instructive to label the edges in the conflict multigraph with the variables causing the conflicts:



If we consider in any conflict multigraph the subgraphs induced by the edges labelled by some given variable, then we obtain a *complete bipartite graph*, called a **biclique**. Any clause-set corresponds to a **biclique decomposition** of its conflict multigraph. In our example it is G decomposed into two bicliques:



Considering *multi*-clause-sets, then also every biclique decomposition of a graph G yields canonically a multi-clause-set F with conflict multigraph G.

Biclique decompositions of graphs are studied in graph theory, and we see, that the study of biclique decompositions of multigraphs is in fact a disguised study of multi-clause-sets:

Every biclique in the decomposition corresponds to one variable of the multi-clause-set, and the multigraph corresponds to the conflict multi-graph of the multi-clause-set.

Using the hermitian rank as complexity measure

By $h(F) \leq n(F)$ one could ask, whether SAT decision for F can be done in time $2^{h(F)}$?!

As a first step Nicola Galesi and I proved, that the class

$$\mathcal{CLS}_h(1) := \{F \in \mathcal{CLS} : h(F) \leq 1\}$$

is interesting, and SAT decision for $\mathcal{CLS}_h(1)$ can be done in polynomial time by applying results from the study of the hypergraph transversal problem.

For more information, have a look at [2, 3, 1].

References

[1] Nicola Galesi and Oliver Kullmann. Polynomial time SAT decision, hypergraph transversals and the hermitian rank. Submitted, 2004.

[2] Oliver Kullmann. On the conflict matrix of clause-sets. Technical Report CSR 7-2003, University of Wales Swansea, Computer Science Report Series, 2003.

[3] Oliver Kullmann. The combinatorics of conflicts between clauses. In *Theory and Applications of Satisfiability Testing 2003*, pages 426–440, 2004.