An Inductive Version of Nash-Williams' Minimal-bad-sequence Argument for Higman's Lemma

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Abstract. Higman's lemma has a very elegant, non-constructive proof due to Nash-Williams [NW63] using the so-called minimal-bad-sequence argument. The objective of the present paper is to give a proof that uses the same combinatorial idea, but is constructive. For a two letter alphabet this was done by Coquand and Fridlender [CF94]. Here we present a proof in a theory of inductive definitions that works for arbitrary decidable well quasiorders.

1 Introduction

This paper is concerned with Higman's lemma [Hig52], usually formulated in terms of well quasi orders.

> If $(A, \leq_A)$ is a well quasiorder, then so is the set A^* of finite sequences in A, together with the embeddability relation $\leq_{A^*}$,

where a sequence $[a_1, \ldots, a_n]$ is embeddable in $[b_1, \ldots, b_m]$ if there is a strictly increasing map $f \colon \{1, \ldots, n\} \to \{1, \ldots, m\}$ such that $a_i \leq_A b_{f(i)}$ for all $i \in \{1, \ldots n\}$.

Among the first proofs of Higman's lemma which all were non-constructive the proof of Nash-Williams using the so-called minimal-bad-sequence argument is considered most elegant. A variant of this proof was translated by Murthy via Friedman's A-translation into a constructive proof [Mur91], however resulting in a huge proof whose computational content couldn't yet be discovered. More direct constructive proofs were given by Schütte/Simpson [SS85], Murthy/Russell [MR90], and Richman/Stolzenberg [RS93]. The Schütte/Simpson proof uses ordinal notations up to ϵ_0 and is related to an earlier proof by Schmidt [Sch79], the other proofs are carried out in a (proof theoretically stronger) theory of inductive definitions. However, their computational content is essentially the same, but does not correspond to that one of Nash-Williams' proof. (The proof theoretic strength of the general minimal-bad-sequence argument is $\Pi^1_1{-}\mathsf{CA}_0$, as

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was shown by Marcone, however it is open whether the special form used for Higman's lemma has the same strength [Mar96].)

The objective of this paper is to present a constructive proof that captures the combinatorial idea behind Nash-Williams' proof. For an alphabet A consisting of two letters this was done by Coquand and Fridlender [CF94]. Their proof can quite easily be extended to a finite alphabet. To obtain a proof for arbitrary decidable well quasiorders, more effort is necessary, as we will describe in section 3.

A proof of Higman's lemma which in contrast to all proofs mentioned above does not require decidability of the given relation $\leq_A$ was given by Fridlender [Fri97]. His proof is based on a proof by Veldman that can be found in [Vel00]. In our formulation of Higman's lemma we will also use an accessibility notion, as it was done in Fridlender's proof.

2 Basic Definitions and an Inductive Characterization of Well Quasiorders

In the whole paper we assume $(A, \leq_A)$ to be a set with a reflexive and transitive, decidable relation.[1]

Definition 1. We use

$a, b, \ldots$	for letters, i.e., elements of a A,
$as, bs, \ldots$	for finite sequences of letters, i.e. elements of A^*,
$v, w, \ldots$	for words, i.e., elements of A^*, [2]
$vs, ws, \ldots$	for finite sequences of words, i.e., elements of A^{**} .

By $as{*}a$ we denote the sequence obtained from the sequence as by appending the element a. $ws{*}w$ is defined similarly. At some places we add brackets to keep the expressions legible. However, unary function application will be written without brackets, in general.

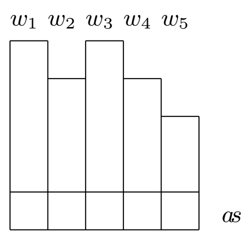
For a finite sequence ws of non-empty words let $\mathsf{lasts}\, ws$ denote the finite sequence consisting of the end-letters of the words of ws, that is,[3]

$$\mathsf{lasts}\,[w_1{*}a_1, \ldots, w_n{*}a_n] = [a_1, \ldots, a_n],\ n \geq 0.$$

[1] Whereas transitivity is only required for historical reasons, but is not used in our proof, decidability plays an essential role.

[2] Although of the same kind we distinguish between finite sequences (of letters) and words, because they will play different rolls, as is illustrated in the picture on the right.

[3] In our picture we have $\mathsf{lasts}\,[w_1, \ldots, w_5] = as$.



Definition 2 (Higman embedding). The embedding relation on A^* can be inductively described by the following rules:

$$\frac{}{[] \leq_{A^*} []} \qquad \frac{v \leq_{A^*} w}{v \leq_{A^*} w{*}a} \qquad \frac{v \leq_{A^*} w,\ a \leq_A b}{v{*}a \leq_{A^*} w{*}b}.$$

Definition 3 (good/bad). A finite sequence $[a_1, \ldots, a_n]$ (respectively an infinite sequence $a_1, a_2, \ldots$) of elements in A is good if there exist $i < j \leq n$ ($i < j < \omega$) such that $a_i \leq_A a_j$; otherwise it is called bad.

Furthermore, we use the notion $\mathsf{good}(as, a)$ if there is an element in as, say the i-th one, such that $(as)_i \leq_A a$. $\mathsf{bad}(as, a)$ stands for $\neg\mathsf{good}(as, a)$.

Finally, $\mathsf{badsubseq}(as)$ determines the first occurring bad subsequence in as:

$$\mathsf{badsubseq}([]) = []$$

$$\mathsf{badsubseq}(as{*}a) = \begin{cases} \mathsf{badsubseq}(as){*}a & \text{if } \mathsf{bad}(\mathsf{badsubseq}(as), a), \\ \mathsf{badsubseq}(as) & \text{otherwise.} \end{cases}$$

Definition 4 (well quasiorder). $(A, \leq_A)$ is a well quasiorder (wqo) if every infinite sequence of elements in A is good.

Definition 5 (the relation $\ll_A$ and its accessible part). The relation $\ll_A \subseteq A^* \times A^*$ is defined by

$$bs \ll_A as :\leftrightarrow bs = as{*}a \text{ for some } a \in A \text{ s.t. } \mathsf{bad}(as, a).$$

The accessible part (also called the well-founded part) of the relation $\ll_A$ is inductively given by the rule

$$\frac{\forall bs \ll_A as \quad \mathsf{acc}_{\ll_A} bs}{\mathsf{acc}_{\ll_A} as}$$

and provides the following induction principle[4] for any formula ϕ:

$$\frac{\forall as.\, \forall bs \ll_A as\ \phi(bs) \to \phi(as)}{\forall as.\, \mathsf{acc}_{\ll_A} as \to \phi(as)}.$$

Definition 6 (well quasiorder, inductive characterization[5]). $(A, \leq_A)$ is a well quasiorder if $\mathsf{acc}_{\ll_A} []$ holds.

Definitions 3 to 6 should be understood for arbitrary (reflexive and transitive) relations, not only for our fixed $(A, \leq_A)$. We will use them also for $(A^*, \leq_{A^*})$. Moreover, the operation acc will also be applied to the relations $\ll_{A^*}$ and $\prec$, still to be defined.

[4] At some places we use a seemingly stronger induction principle where the premise is of the form $\forall as. \mathsf{acc}_{\ll_A} as \to \forall bs \ll_A as\ \phi(bs) \to \phi(as)$. This principle can be easily derived from the above one.

[5] In our paper we only deal with this second definition of a well quasiorder since it is very suitable for a constructive proof. For sake of completeness we give an argument for the equivalence of definition 4 and definition 6. To prove that definition 6 implies definition 4 one shows more generally that for all as such that $\mathsf{acc}_{\ll_A} as$ holds every infinite sequence, starting with as, is good. The reverse direction is an instance of Brouwer's axiom of bar induction.

3 Towards a Constructive Proof

In order to motivate further definitions we first want to give the idea behind the constructive proof. This is best done by showing the connection between the classical and the constructive proof. To this end we shortly recall Nash-Williams' minimal-bad-sequence proof and show how the main steps are captured by the inductive proof. We also include an informal idea of the latter.

The steps of the Nash-Williams' proof:

1. In order to show "wqo(A) implies wqo(A^*)", assume for contraction that there is a bad sequence of words.
2. Among all infinite bad sequences we choose (using classical dependent choice) a minimal bad sequence, i.e., a sequence, say $(w_i)_{i<\omega}$, which is minimal with respect to a lexicographical order on infinite sequences of words (where w_1 is less or equal w_2, if w_1 is an initial segment of w_2).
3. Since $w_i \neq [\,]$, let $w_i = v_i{*}a_i$ for all i. Using Ramsey's theorem and the fact that our alphabet A is a well quasiorder, we know that there exists an infinite subsequence $a_{\kappa_1} \leq_A a_{\kappa_2} \leq_A \cdots$ of the sequence $(a_i)_{i<\omega}$. This also determines a corresponding sequence $w_1, \ldots, w_{\kappa_1-1}, v_{\kappa_1}, v_{\kappa_2}, \ldots$.
4. The sequence $w_1, \ldots, w_{\kappa_1-1}, v_{\kappa_1}, v_{\kappa_2}, \ldots$. must be bad (otherwise $(w_i)_{i<\omega}$ would be good), but this contradicts the minimality in 2.

In the constructive proof this steps correspond to

1. Prove inductively "$\mathsf{acc}_{\ll_A} [\,] \to \mathsf{acc}_{\ll_{A^*}} [\,]$".
2. The minimality argument will be replaced by structural induction on words.
3. Given a bad sequence $ws = [w_1, \ldots, w_n]$ s.t. $w_i = v_i{*}a_i$, we are interested in all subsequences $a_{\kappa_1} \leq_A \cdots \leq_A a_{\kappa_l}$ of maximal length[6] and their corresponding sequences $w_1, \ldots, w_{\kappa_1-1}, v_{\kappa_1}, \ldots, v_{\kappa_l}$. In the proof these sequences will be computed by the procedure forest which takes ws as input and yields a forest labeled by pairs in $A^{**} \times A$. In the produced forest the right-hand components of each path form a weakly ascending subsequence of $[a_1, \ldots, a_n]$ and the corresponding sequence of form $w_1, \ldots, w_{\kappa_1-1}, v_{\kappa_1}, \ldots, v_{\kappa_l}$ could be read off in the left-hand component of the endnode of such a path. If we extend the sequence ws badly by a word $v{*}a$, then in the existing forest either new nodes, possibly at several places, are inserted, or a new singleton tree with node $\langle ws{*}v, a\rangle$ is added. Now the informal idea of the inductive proof is: if in $\mathsf{forest}\, ws$ not infinitely often new nodes could be inserted and if also not infinitely often new trees could be added, then ws could not be extended badly infinitely often. Formally this will be captured by the statement: $\forall ws.\, \mathsf{acc}_{\ll_A} \mathsf{badsubseq}(\mathsf{lasts}\, ws) \to \mathsf{acc}_{\prec} \mathsf{forest}\, ws \to \mathsf{acc}_{\ll_{A^*}} ws$
4. The first part of item 4 corresponds to lemma 1.

[6] By maximal length we mean that we only look at those subsequences which are ascending, but not contained in other ones, for instance our chosen subsequences of $[1, 4, 3, 0, 3]$ are $[1, 4]$, $[1, 3, 3]$ and $[0, 3]$.

We proceed with the formal definition of forest and the relation $\prec$ on forests.

Definition 7. We use

t	for elements in $T(A^{**} \times A)$, i.e., trees labeled by pairs in $A^{**} \times A$,
f, ts	for elements in $(T(A^{**} \times A))^*$, i.e., forests.

The tree with root $\langle ws, a\rangle$ and finite sequence of immediate subtrees ts is written $\langle ws, a\rangle ts$. We use the destructors left and right for pairs and the destructors root and subtrees for trees, hence $\mathsf{root}\,\langle ws, a\rangle ts = \langle ws, a\rangle$ and $\mathsf{subtrees}\,\langle ws, a\rangle ts = ts$. For better readability we set:

$$\begin{aligned}
\mathsf{newtree}\,\langle ws, a\rangle &:= \langle ws, a\rangle[\,],\\
\mathsf{roots}\,[t_1, \dots, t_n] &:= [\mathsf{root}\,t_1, \dots, \mathsf{root}\,t_n],\\
\mathsf{lefts}\,[\langle vs_1, a_1\rangle, \dots, \langle vs_n, a_n\rangle] &:= [vs_1, \dots, vs_n],\\
\mathsf{rights}\,[\langle vs_1, a_1\rangle, \dots, \langle vs_n, a_n\rangle] &:= [a_1, \dots, a_n].
\end{aligned}$$

Definition 8. Let $ws \in A^{**}$ be a sequence of non-empty words. Then $\mathsf{forest}\,ws \in T((A^{**} \times A))^*$ is defined recursively by: [7] [8]

$$\begin{aligned}
\mathsf{forest}\,[\,] &= [\,],\\
\mathsf{forest}\,ws{*}(w{*}a) &= \begin{cases} \mathsf{insertforest}(\mathsf{forest}\,ws, w, a) & \text{if } \mathsf{good}(\mathsf{badsubseq}(\mathsf{lasts}\,ws), a)\\ (\mathsf{forest}\,ws) * \mathsf{newtree}\,\langle ws{*}w, a\rangle & \text{otherwise,}\end{cases}
\end{aligned}$$

where

$$\mathsf{insertforest}(f, w, a) = \mathrm{map}\left(\lambda t \begin{bmatrix} \text{if } & \mathsf{right}\,(\mathsf{root}\,t) \leq_A a\\ & \mathsf{inserttree}(t, w, a)\\ & t \end{bmatrix}\right) f$$

and

$$\mathsf{inserttree}(\langle vs, a'\rangle ts, w, a) =$$
$$\begin{cases} \langle vs, a'\rangle\mathsf{insertforest}(ts, w, a) & \text{if } \mathsf{good}(\mathsf{rights}\,(\mathsf{roots}\,ts), a),\\ \langle vs, a'\rangle(ts * \mathsf{newtree}\,\langle vs{*}w, a\rangle) & \text{otherwise.}\end{cases}$$

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[7] For sake of simplicity we define insertforest by a map operation, i.e., we insert a new node at every possible place. However, it would already suffice to insert at least once and it even could be arbitrarily chosen where to insert.

[8] In case of a finite alphabet $\mathsf{forest}\,ws$ has only non-branching trees where the right-hand components of such a tree are constant. So, if in the notion of [CF94] we have $T_0(vs, ws)$ or $R_0(vs, ws)$, in our setting the sequence vs could be read off as the left-hand component in the endnode of the tree whose right-hand components are 0.

Definition 9. Let f and f' be forests in $T((A^{**} \times A))^*$. Then

$$f' \prec f :\leftrightarrow \begin{cases} f' = \mathsf{insertforest}(f, w, a) \text{ for some } w \in A^*, a \in A \\ \text{such that } f' \neq f \text{ and the left-hand component of} \\ \text{each label in } f' \text{ is a bad sequence in } A^{**}. \end{cases}$$

Lemma 1. Let ws be a bad sequence of non-empty words. Then in every label of $\mathsf{forest}\, ws$ the left-hand component is a bad sequence.

Proof. IND(structure of ws). 1. $ws = []$. Clear.

2. Assume that every left-hand component of a label in $\mathsf{forest}\, ws$ is bad and look at the nodes in $\mathsf{forest}\, ws{*}(w{*}a)$ where $ws{*}(w{*}a)$ is assumed to be bad.

Case 1: $\mathsf{bad}(\mathsf{badsubseq}(\mathsf{lasts}\, ws), a)$. Then in $\mathsf{forest}\, ws{*}(w{*}a)$ only one node was added, i.e., the node with label $\langle ws{*}w, a\rangle$ where by assumption $ws{*}w$ is bad.

Case 2: $\mathsf{good}(\mathsf{badsubseq}(\mathsf{lasts}\, ws), a)$. In this case[9] some nodes of the form $\langle vs{*}w, a\rangle$ were inserted in $\mathsf{forest}\, ws$ where vs is a left-hand component of a node in $\mathsf{forest}\, ws$ which by assumption is bad. Assume $\mathsf{good}(vs, w)$, that is, $\exists i (vs)_i \leq_{A^*} w$ and show $\bot$.

Case 2.1: $(vs)_i$ is a word in ws. Then, by the Higman embedding we obtain $(vs)_i \leq_{A^*} w{*}a$ – contradicting the badness of $ws{*}(w{*}a)$.

Case 2.2: $(vs)_i$ is a word in ws cut by an end letter a_0 and by the construction of the forests it holds $a_0 \leq_A a$.[10] Then, again by the Higman embedding it follows $(vs)_i{*}a_0 \leq_{A^*} w{*}a$. Contradiction. □

Lemma 2. i) $\mathsf{acc}_\prec\, []$. ii) $\mathsf{acc}_\prec\, f \wedge \mathsf{acc}_\prec\, [t] \rightarrow \mathsf{acc}_\prec\, f * t$.

Proof. i) $\mathsf{acc}_\prec\, []$ holds by definition, since there is no tree in which new nodes could be inserted. ii) Clear, since insertforest is defined by a map-operation. □

Lemma 3. Assume $\mathsf{acc}_{\ll_A}\, []$. Then $\forall ws.\, \mathsf{acc}_{\ll_{A^*}}\, ws \rightarrow \forall a.\, \mathsf{acc}_\prec\, [\mathsf{newtree}\, \langle ws, a\rangle]$.

Proof. $\mathsf{IND}_1(\mathsf{acc}_{\ll_{A^*}})$: $\mathsf{IH}_1 : \forall vs \ll_{A^*} ws, \forall a.\, \mathsf{acc}_\prec\, [\mathsf{newtree}\, \langle vs, a\rangle]$. Let $a \in A$. Instead of proving $\mathsf{acc}_\prec\, [\mathsf{newtree}\, \langle ws, a\rangle]$ we show more generally that this assertion holds for all t with $\mathsf{root}\, t = \langle ws, a\rangle$ such that

(a) the subtrees of t form a forest in $\mathsf{acc}_\prec$ and

(b) $\mathsf{rights}\, (\mathsf{roots}\, (\mathsf{subtrees}\, t))$ is sequence in $\mathsf{acc}_{\ll_A}$.[11]

We do this by main induction on (b) and side induction on (a), i.e., formally we prove

$$\begin{aligned} &\forall as.\ \mathsf{acc}_{\ll_A}\, as \rightarrow \\ &\forall ts.\ \mathsf{acc}_\prec\, ts \rightarrow \\ &\forall t.\ \ \mathsf{root}\, t = (ws, a) \wedge \mathsf{subtrees}\, t = ts \wedge as = \mathsf{rights}\, (\mathsf{roots}\, (\mathsf{subtrees}\, t)) \rightarrow \\ &\qquad \mathsf{acc}_\prec\, [t]. \end{aligned}$$

[9] Here, we only sketch the combinatorial part of the proof; a formal proof involves an induction on the tree structure.

[10] Note that transitivity is not required.

[11] It's intended that $[t]$ lies in the image of the partial function forest, however we don't need this restriction in the formulation of the lemma.

$\mathsf{IND}_2(\mathsf{acc}_{\ll_A})$. Assume that we have an as such that $\mathsf{acc}_{\ll_A}\, as$.

$\mathsf{IND}_3(\mathsf{acc}_{\prec})$. Let ts be such that $\mathsf{acc}_{\prec}\, ts$ and fix t such that $\mathsf{root}\, t = \langle ws, a\rangle$, $\mathsf{subtrees}\, t = ts$, and $as = \mathsf{rights}\,(\mathsf{roots}\,(\mathsf{subtrees}\, t))$.

We have to prove $\mathsf{acc}_{\prec}\, [t]$. By the definition of $\mathsf{acc}_{\prec}$ and $\prec$ if suffices to show $\forall t'.\, [t'] \prec [t] \to \mathsf{acc}_{\prec}\, [t']$ where $t' = \mathsf{inserttree}(t, w, a') \neq t$ for some $w \in A^*, a' \in A$ and all left-hand components of nodes in t' are required to be bad. We prove the assertion by case distinction on the definition of $\mathsf{inserttree}$.

Case 1: $t' = \langle ws, a\rangle\, (ts * \mathsf{newtree}\,\langle ws{*}w, a'\rangle)$ for some w and a' such that $\mathsf{bad}(as, a')$. Then we have

$$\begin{aligned} \mathsf{root}\, t' &= \langle ws, a\rangle, \\ \mathsf{subtrees}\, t' &= ts * \mathsf{newtree}\,\langle ws{*}w, a'\rangle, \\ as{*}a' &= \mathsf{rights}\,(\mathsf{roots}\,(ts * \mathsf{newtree}\,\langle ws{*}w, a'\rangle)). \end{aligned}$$

Since all left-hand components in t' are supposed to be bad, in particular, we have that $ws{*}w$ is bad, i.e., $ws{*}w \ll_{A^*} ws$. By IH_1 we obtain $\mathsf{acc}_{\prec}\, [\mathsf{newtree}\,\langle ws{*}w, a'\rangle]$, and hence by lemma 2

$$\mathsf{acc}_{\prec}\, ts * \mathsf{newtree}\,\langle ws{*}w, a'\rangle.$$

Now, since $as{*}a' \ll_A as$, we may apply IH_2 to $as{*}a', ts * \mathsf{newtree}\,\langle ws{*}w, a'\rangle$ and t' and conclude $\mathsf{acc}_{\prec}\, [t']$.

Case 2: $t' = \langle ws, a\rangle\, \mathsf{insertforest}(ts, w, a')$ where a' such that $\mathsf{good}(as, a')$. In this case we have

$$\begin{aligned} \mathsf{root}\, t' &= \langle ws, a\rangle, \\ \mathsf{subtrees}\, t' &= \mathsf{insertforest}(ts, w, a'), \\ as &= \mathsf{rights}\,(\mathsf{roots}\,(\mathsf{subtrees}\, t')). \end{aligned}$$

Moreover $[t'] \prec [t]$ implies $\mathsf{subtrees}\, t' \prec \mathsf{subtrees}\, t = ts$, and by IH_3, applied to $\mathsf{subtrees}\, t'$ and t', we end up with $\mathsf{acc}_{\prec}\, [t']$.

Now, the proof of the general assertion is completed, and we may put $as = []$, $f = []$ and $t = \mathsf{newtree}\,\langle ws, a\rangle$. Since we have $\mathsf{acc}_{\ll_A}\, []$ by assumption and $\mathsf{acc}_{\prec}\, []$ by lemma 2, we obtain $\mathsf{acc}_{\ll_{A^*}}\, ws \to \mathsf{acc}_{\prec}\, [\mathsf{newtree}\,\langle ws, a\rangle]$. □

4 The Proof of Higman's Lemma

Proposition 1 (Higman's Lemma). $\mathsf{acc}_{\ll_A}\, [] \to \mathsf{acc}_{\ll_{A^*}}\, []$.

Proof. Assume $\mathsf{acc}_{\ll_A}\, []$. We show more generally

$$\begin{aligned} &\forall as.\ \mathsf{acc}_{\ll_A}\, as \to \\ &\forall f.\ \ \mathsf{acc}_{\prec}\, f \to \\ &\forall ws.\ as = \mathsf{badsubseq}(\mathsf{lasts}\, ws) \ \wedge\ f = \mathsf{forest}\, ws \to \mathsf{acc}_{\ll_{A^*}}\, ws. \end{aligned}$$

$\mathsf{IND}_1(\mathsf{acc}_{\ll_A})$. Let as be such that $\mathsf{acc}_{\ll_A}\, as$ and $\mathsf{IH}_1 : \forall bs \ll_A as, \forall f.\, \mathsf{acc}_{\prec}\, f \to \forall ws.\, bs = \mathsf{badsubseq}(\mathsf{lasts}\, ws) \ \wedge\ f = \mathsf{forest}\, ws \to \mathsf{acc}_{\ll_{A^*}}\, ws$.

$\mathsf{IND}_2(\mathsf{acc}_\prec)$. Let f be s.t $\mathsf{acc}_\prec f$ and $\mathsf{IH}_2 : \forall f' \prec f, \forall ws.\ \mathsf{badsubseq}(\mathsf{lasts}\, ws) = as \wedge f' = \mathsf{forest}\, ws \to \mathsf{acc}_{\ll_{A^*}} ws$ and assume that we have ws such that $as = \mathsf{badsubseq}(\mathsf{lasts}\, ws)$ and $f = \mathsf{forest}\, ws$. In order to prove $\mathsf{acc}_{\ll_{A^*}} ws$ we fix w s.t. $ws{*}w$ is bad and show $\mathsf{acc}_{\ll_{A^*}} ws{*}w$ by induction on the structure of w:

$\mathsf{IND}_3(w)$. 1. $\mathsf{acc}_{\ll_{A^*}} ws[]$ holds by definition of $\mathsf{acc}_{\ll_{A^*}}$.

2. Now, assume that we have a word of form $w{*}a$. We show $\mathsf{acc}_{\ll_{A^*}} ws{*}(w{*}a)$ by case analysis on whether or not $\mathsf{bad}(as, a)$.

Case 2.1: $\mathsf{bad}(as, a)$. Then we have

$$\begin{aligned} as{*}a &= \mathsf{badsubseq}(\mathsf{lasts}\,(ws{*}(w{*}a))),\\ f * \mathsf{newtree}\,\langle ws{*}w, a\rangle &= \mathsf{forest}\,(ws{*}(w{*}a)). \end{aligned}$$

First, we show $\mathsf{acc}_\prec f * \mathsf{newtree}\,\langle ws{*}w, a\rangle$. By assumption we already have $\mathsf{acc}_\prec f$ and by IH_3 $\mathsf{acc}_{\ll_{A^*}} ws{*}w$. Hence, by lemma 3 we obtain $\mathsf{acc}_\prec [\mathsf{newtree}\,\langle ws{*}w, a\rangle]$ and by lemma 2 we may conclude

$$\mathsf{acc}_\prec f * \mathsf{newtree}\,\langle ws{*}w, a\rangle.$$

Now we are able to apply IH_1 (to $as{*}a$, $f * \mathsf{newtree}\,\langle ws{*}w, a\rangle$ and $ws{*}(w{*}a)$) and end up with $\mathsf{acc}_{\ll_{A^*}} ws{*}(w{*}a)$.

Case 2.2: $\mathsf{good}(as, a)$. In this case it follows

$$\begin{aligned} as &= \mathsf{badsubseq}(\mathsf{lasts}\,(ws{*}(w{*}a))),\\ \mathsf{insertforest}(f, w, a) &= \mathsf{forest}\,(ws{*}(w{*}a)). \end{aligned}$$

By lemma 1 all left-hand components of nodes in $\mathsf{insertforest}(f, w, a)$ are bad. Moreover, $\mathsf{insertforest}(f, w, a) \neq f$ since $\mathsf{good}(as, a)$ and $\mathsf{badsubseq}(\mathsf{lasts}\, ws) = as = \mathsf{rights}\,(\mathsf{roots}\,(\mathsf{forest}\, ws))$ imply that indeed at least one node was inserted. Hence, we obtain

$$\mathsf{insertforest}(f, w, a) \prec f$$

and we may apply IH_2 (to $\mathsf{insertforest}(f, w, a)$ and $ws{*}(w{*}a)$) and conclude $\mathsf{acc}_{\ll_{A^*}} ws{*}(w{*}a)$.

This completes the proof of the general assertion. Now, by putting $as = [\,], f = [\,]$ and $ws = [\,]$ and the fact that $\mathsf{acc}_\prec [\,]$ holds by definition we obtain $\mathsf{acc}_{\ll_A} [\,] \to \mathsf{acc}_{\ll_{A^*}} [\,]$. □

5 Conclusion

We presented a new constructive proof of Higman's lemma for arbitrary decidable well quasiorders in a theory of inductive definitions. We hope not only that this proof gives more insight in the interplay of classical proofs using a minimal bad sequence argument and constructive proofs, but also that this strategy is extendible to other non-constructive theorems, for instance Kruskal's tree theorem and the so-called extended Kruskal theorem, also known as Kruskal's theorem with gap condition. Both have proofs using a minimal-bad-sequence argument (see [NW63] resp. [Sim85]), however no constructive proof at all is known for the latter. Kruskal's theorem was proved constructively (see [RW93] for a proof using ordinal notations or [Sei01] for an inductive reformulation of this proof, and [Vel00] for a proof not requiring decidability). These proofs, however, are quite involved in comparison with the minimal-bad-sequence proof.

We do not claim that our proof of Higman's lemma is 'better' than the other constructive proofs mentioned in the introduction, but, as already stated, it uses a different combinatorial idea, hence results in another algorithm. An analysis of these different algorithms is still missing and could give rise to an interesting case study in machine supported theorem proving.

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