Kruskal's tree theorem in a constructive theory of inductive definitions

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Abstract

We give a constructive proof of Kruskal's tree theorem by using an inductive characterization of well quasi orders.

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1 Introduction

This paper is about a famous theorem in infinitary combinatorics, Kruskal's tree theorem, in a context of constructive mathematics. Usually, Kruskal's theorem is formulated in terms of well quasi orders:

> If $(A, \leq)$ is a well quasi order, then so is $T(A)$, the set of finite trees with labels in A, together with the embeddability relation.

Here a set A with a reflexive and transitive relation $\leq$ is a well quasi order if for every infinite sequence $(a_i)_{i<\omega}$ there exist indices $i<j<\omega$ such that $a_i \leq a_j$. A tree is embeddable into another one if there exists a one to one map on them respecting infima of nodes such that the label of each node is less or equal to that of its image.

Kruskal's theorem [Kru60] has a very elegant classical proof due to Nash-Williams [NW63] using a minimal-bad-sequence-argument. This proof is often referred to in term rewriting, the main field of application where Kruskal's theorem implies termination of simplification orders.

To obtain a constructive proof we reformulate Kruskal's Theorem. The main idea is to use an inductive definition of the notion of a well quasi order: call a finite sequence $[a_1, \ldots, a_n]$ bad if $a_i \not\leq a_j$ for all $i < j \leq n$. Intuitively, being a well quasi order means that no bad sequence can be badly extended infinitely often. In other words: the extension relation between finite bad sequences (denoted $\ll$) is wellfounded, where by 'wellfounded' we mean that the induction principle for this relation holds. Technically this is expressed by the statement that all bad sequences are in the accessible part of the relation $\ll$. In fact, this already is the case if the empty sequence is in the accessible part, i.e $\mathsf{acc}_{\ll}\,[]$. Hence, the inductive formulation of Kruskal's Theorem is

$$\mathsf{acc}_{\ll_A}\,[] \to \mathsf{acc}_{\ll_{T(A)}}\,[].$$

Constructively the inductive characterization of a well quasi order implies the classical one. The converse is an instance of Brouwer's axiom of bar induction. However, since we prove the stronger inductive formulation, in our approach we don't make use of Brouwer's axiom.

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The idea of reformulating Kruskal's Theorem in order to get a constructive proof can also be found in the proofs of Schmidt [Sch79], Rathjen and Weiermann [RW93], and Hasegawa [Has94] using (ordinal) notation systems. Schmidt gave a proof of Kruskal's theorem by determining the maximal order type of the set of labeled trees (i.e. the supremum of the order types of all well orders the quasi order of trees can be extended to). Rathjen and Weiermann [RW93] reorganized this proof such that its constructive content became more visible and thereby solved the problem of determining the proof theoretic strength of Kruskal's theorem (see [Sim85] or [Gal91]). Hasegawa's proof [Has94] is similar to that of [RW93]. He uses a different well ordering but doesn't carry out a constructive well ordering proof. Our proof follows the combinatorial idea of [RW93]. However, using the inductive characterization we do not need to introduce ordinal notations, nor do we have to refer to other results. Although almost completely formalized, this proof is short and intuitive and, as inductive definitions are available in most theorem provers, could serve as an interactive proof script.

The inductive characterization of a well quasi order was introduced by Martin-Löf [Mar70]. Coquand [Coq93] used it for a short proof of Higman's Lemma (on a two element alphabet), a specialization of Kruskal's theorem to sequences. Using essentially the same method, Richman and Stolzenberg [RS93] proved Higman's lemma for decidable quasi orders. A proof of Higman's lemma not requiring decidability was given by Fridlender [Fri97], inspired by ideas of Veldman. Our proof of Kruskal's theorem requires decidable relations, an analogous result for arbitrary relations would be highly desirable.

Many other theorems typically known to hold classically can be reformulated by means of inductive definitions thus admitting constructive proofs. Recent applications in infinitary combinatorics and algebra were given by Coquand, Persson [CP99], [Per99] and Fridlender [Fri99].

2 Well quasi orders

In this section we introduce the formal inductive definition of a well quasi order and prove two simple properties of it.

Conventions. In the following let A be a decidable quasi order, i.e. a set with a binary, decidable, reflexive, and transitive[1] relation $\leq_A$. $A^* := \{[a_1, \dots, a_n] : n \in \mathbb{N}, a_i \in A\}$. We use

a, b	for elements of A (letters)
as, bs	for elements of A^* (words, i.e. finite sequences of letters)

Let $as = [a_1, \dots, a_n]$. We write $as * a$ for the sequence $[a_1, \dots, a_n, a]$ and $as@bs$ for the concatenation of two sequences as and bs in A^*. $[]$ is the empty sequence. If $i \leq \text{length } as$, the i-th element of as is denoted by $(as)_i$.

Definition. We call a finite sequence $[a_1, \dots, a_n]$ good if there exists $i < j \leq n$ such that $a_i \leq_A a_j$, otherwise bad. Let $\text{Bad}(A)$ be the set of all finite bad sequences in A. Let $as = [a_1, \dots, a_n] \in \text{Bad}(A)$. We set

$$A_{as} := \{b \in A : \forall i \leq n.\, a_i \not\leq_A b\},$$

$\leq_{A_{as}}$ is the relation $\leq_A$ restricted to the set A_{as}. Moreover, we define a relation $\ll_A \subseteq \text{Bad}(A) \times \text{Bad}(A)$ by

$$bs \ll_A as :\leftrightarrow bs = as * a \text{ for some } a \in A_{as}.$$

[1] For historical reasons we have formulated Kruskal's theorem for quasi orders. Therefore we assume $\leq_A$ to be transitive, although transitivity is not used in the proof.

The accessible part (also called the wellfounded part) of the relation $\ll_A$ is inductively given by the rule

$$\frac{\forall bs \ll_A as \quad \mathsf{acc}_{\ll_A} bs}{\mathsf{acc}_{\ll_A} as}.$$

According to the definition of $\mathsf{acc}_{\ll_A}$ the following induction principle holds for any formula $\phi(as)$:

$$\frac{\forall as.\, \forall bs \ll_A as\ \phi(bs) \to \phi(as)}{\forall as.\, \mathsf{acc}_{\ll_A} as \to \phi(as)}.$$

Analogously, the acc-notion will be used for other relations. Now, $(A, \leq_A)$ is a well quasi order iff $\ll_A$ is wellfounded on $\mathrm{Bad}(A)$, i.e. if all bad sequences are in the accessible part of the relation $\ll_A$.

The remark below shows that being a well quasi order (as defined in the introduction) is equivalent to $\mathsf{acc}_{\ll_A} []$.

Remark. $\mathsf{acc}_{\ll_A} [] \iff A$ is a well quasi order.

Proof. "$\Longrightarrow$". We show more generally

$$\forall as.\, \mathsf{acc}_{\ll_A} as \to \forall (a_i)_{i<\omega}, \forall n.\, [a_1, \ldots, a_n] = as \to \exists i < j.\, a_i \leq_A a_j$$

by induction on $\mathsf{acc}_{\ll_A}$. The statement will follow by $as = []$. Let $as \in \mathrm{Bad}(A)$ and assume the induction hypothesis:

$$\forall bs \ll_A as, \forall (a_i)_{i<\omega}, \forall n.\, [a_1, \ldots, a_n] = bs \to \exists i < j.\, a_i \leq_A a_j.$$

Now, let an infinite sequence $(a_i)_{i<\omega}$ and n such that $[a_1, \ldots, a_n] = as$ be given. Case 1: there exists an $i \leq n$ such that $a_i \leq_A a_{n+1}$. Then we are done. Case 2: $\forall i \leq n\, a_i \not\leq_A a_{n+1}$, i.e. $as * a_{n+1} \ll_A as$. By induction hypothesis we obtain i and j such that $a_i \leq_A a_j$ and we have also proved our assertion.

"$\Longleftarrow$". This direction is an instance of Brouwer's axiom of bar induction which is intuitionistically considered acceptable. We also give a classical proof: assume that $\mathsf{acc}_{\ll_A} []$ does not hold. Then, by the definition of $\mathsf{acc}_{\ll_A}$ there is an element $a_1 \in A$ such that $\neg\mathsf{acc}_{\ll_A} [a_1]$. By iteration (using the Axiom of Dependent Choice) we get a sequence $(a_i)_{i<\omega}$, such that for all n holds: $\neg\mathsf{acc}_{\ll_A} [a_1, \ldots, a_n]$ and $\forall i < j \leq n\, a_i \not\leq_A a_j$, contradicting A being a well quasi order. ⋄

Lemma 1. $(\forall a \in A\, \mathsf{acc}_{\ll_{A_{[a]}}} []) \to \mathsf{acc}_{\ll_A} []$.

Proof. Assume $\forall a \in A\, \mathsf{acc}_{\ll_{A_{[a]}}} []$. By definition of $\mathsf{acc}_{\ll_A}$ we need to show $\forall a \in A\, \mathsf{acc}_{\ll_A} [a]$. Let $a \in A$. By assumption we have $\mathsf{acc}_{\ll_{A_{[a]}}} []$. Then $\mathsf{acc}_{\ll_A} [a]$ follows from the following more general assertion, if we set $as = [a]$ and $bs = []$:

$$\forall as \in \mathrm{Bad}(A), \forall bs \in \mathrm{Bad}(A_{as}).\, \mathsf{acc}_{\ll_{A_{as}}} bs \to \mathsf{acc}_{\ll_A} as@bs.$$

$\mathsf{Ind}(\mathsf{acc}_{\ll_{A_{as}}})$. Fix as, bs and assume $\mathsf{ih} : \forall bs' \ll_{A_{as}} bs\, \mathsf{acc}_{\ll_A} as@bs'$. Again by definition of $\mathsf{acc}_{\ll_A}$ it suffices to show $\mathsf{acc}_{\ll_A} as@bs * c$ for an arbitrary $c \in A_{as@bs}$ and since $bs * c \ll_{A_{as}} bs$ we are done by induction hypothesis. ⋄

Definition. A quasi embedding from $(A, \leq_A)$ to $(B, \leq_B)$ is an injective map $f\colon A \to B$, that doesn' t create any new $\leq$-relations; i.e. for all $a_1, a_2 \in A$:

$$f(a_1) \leq_B f(a_2) \to a_1 \leq_A a_2.$$

Lemma 2. Let $f: A \to B$ be a quasi embedding. Then $\mathsf{acc}_{\ll_B}[] \to \mathsf{acc}_{\ll_A}[]$.

Proof. We show

$$\forall bs.\, \mathsf{acc}_{\ll_B}\, bs \to \forall as.\, f(as) = bs \to \mathsf{acc}_{\ll_A}\, as,$$

where $f(as)$ means that f is applied component wise to as, i.e. $f([a_1, \ldots, a_n]) = [f(a_1), \ldots, f(a_n)]$. Then $\mathsf{acc}_{\ll_B}[] \to \mathsf{acc}_{\ll_A}[]$ follows by $as = bs = []$.
$\mathsf{Ind}(\mathsf{acc}_{\ll_B})$. Fix bs and assume

$$\mathsf{ih} : \forall bs' \ll_B bs, \forall as.\, f(as) = bs' \to \mathsf{acc}_{\ll_A}\, as.$$

Let as s.t. $f(as) = bs$. We have to show $\mathsf{acc}_{\ll_A}\, as$, i.e. $\forall a \in A_{as}\, \mathsf{acc}_{\ll_A}\, as * a$. $a \in A_{as}$ satisfies $(as)_i \not\leq_A a$ for all $i \leq$ length as and, since f is a quasi embedding, it follows $f((as)_i) \not\leq_B f(a)$, i.e. $(bs)_i \not\leq_B f(a)$. Hence $bs * f(a) \ll_B bs$, and by ih, applied to $bs * f(a)$ and $as * a$ we obtain $\mathsf{acc}_{\ll_A}\, as * a$. ◇

3 Kruskal's tree theorem

Now, we are going to state and prove the inductive formulation of Kruskal's theorem. We start with a formal definition of $(T(A), \leq_{T(A)})$ and then introduce an additional structure on trees due to [Sch79] which will be essential for the combinatorial idea of the proof.

Definition. The set $T(A)$ of finite trees with labels in A is inductively defined by

$$\frac{a \in A \qquad t_1, \ldots, t_n \in T(A)}{a\, t_1 \ldots t_n \in T(A),}$$

where $a\, t_1 \ldots t_n$, $0 \leq n$, is the tree consisting of a root with label a and the subtrees $t_1, \ldots, t_n$. Moreover, we define the embeddability relation $\leq_{T(A)}$ inductively by the following rules:

$$\frac{t \leq_{T(A)} u_j, \quad \text{for some } j \leq m}{t \leq_{T(A)} a\, u_1 \ldots u_m} \qquad \frac{a \leq_A b \qquad [t_1, \ldots, t_n] \leq_{T(A)^*} [u_1, \ldots, u_m]}{a\, t_1 \ldots t_n \leq_{T(A)} b\, u_1 \ldots u_m}.$$

Here $\leq_{T(A)^*}$ refers to the Higman embedding. $\leq_{A^*}$ on A^* is inductively defined by

$$\frac{}{[] \leq_{A^*} []} \qquad \frac{as \leq_{A^*} bs}{as \leq_{A^*} bs * b} \qquad \frac{as \leq_{A^*} bs \quad a \leq_A b}{as * a \leq_{A^*} bs * b}.$$

Definition. Given quasi orders $(A_1, \leq_{A_1}), \ldots, (A_n, \leq_{A_n})$ we define disjoint union $\dot\cup\{A_i : i \leq n\}$ and cartesian product $\times\{A_i : i \leq n\}$ via

$$a \leq_{\dot\cup A_i} a' \quad \leftrightarrow \quad a, a' \in A_i \wedge a \leq_{A_i} a' \text{ for some } i \leq n,$$
$$(a_1, \ldots, a_n) \leq_{\times A_i} (a'_1, \ldots, a'_n) \quad \leftrightarrow \quad a_i \leq_{A_i} a'_i \text{ for all } i \leq n$$

Definition. Let $(A_0, \leq_{A_0}), \ldots, (A_n, \leq_{A_n})$ and $0 < \alpha_0 < \cdots < \alpha_n \leq \omega$ be given. By $T\binom{A_n \ldots A_0}{\alpha_n \ldots \alpha_0}$ we denote the set of all trees with labels in $\dot\cup\{A_i : i \leq n\}$ such that every node with a label in A_i has less than α_i immediate successors. The embeddability relation on $T\binom{A_n \ldots A_0}{\alpha_n \ldots \alpha_0}$ is the restriction of $\leq_{T(\dot\cup\{A_i : i \leq n\})}$ on $T\binom{A_n \ldots A_0}{\alpha_n \ldots \alpha_0}$.

Kruskal's tree theorem. $\mathsf{acc}_{\ll_A}[] \to \mathsf{acc}_{\ll_{T\binom{A}{\omega}}}[]$.

We first give a rough idea of the proof, in order to motivate further definitions. To show $\mathsf{acc}_{\ll_{T\binom{A}{\omega}}}[]$, because of lemma 1, it suffices to show $\mathsf{acc}_{\ll_{T\binom{A}{\omega}_{[t]}}}[]$ for an arbitrary tree t. If there is a quasi embedding of $T\binom{A}{\omega}_{[t]}$ into an appropriate tree set $T\binom{B\ A_{[a]}}{n\ \ \omega}$ and if $\mathsf{acc}_{\ll_{T\binom{B\ A_{[a]}}{n\ \ \omega}}}[]$ holds we are done by lemma 2. Now, $\binom{B\ A_{[a]}}{n\ \ \omega}$ is in a certain sense lexicographically smaller than $\binom{A}{\omega}$. So we will be finished by induction on this order once we have defined it. However, such a definition would require quantification over sets. Since in this definition we only need sets B of a certain structure, we can restrict ourselves to elements of the set of cartesian compositions of A, a set of names denoted $\mathsf{Cart}(A)$, thus avoiding second order quantification. We define a relation $<$ on $\mathsf{Cart}(A)$ such that for all cartesian compositions $\mathcal{X}$ $\mathsf{acc}_{<_{\mathsf{Cart}(A)}} \mathcal{X}$ implies $\mathsf{acc}_{\ll_X}[]$. In a second step we define the relation $<_{\mathsf{Lex}}$ and show that all cartesian compositions $\mathcal{X}$ are in the accessible part of $<_{\mathsf{Cart}(A)}$, i.e. $\mathsf{acc}_{<_{\mathsf{Cart}(A)}} \mathcal{X}$ holds. The definition of $\mathsf{Cart}(A)$ as well as the notion $\mathcal{X}^x$ and lemma 3 are due to [RW93]. In [Has94] the set $\mathsf{Cart}(A)$ corresponds, with a small restriction, to the class of algebras.

Definition (Cartesian compositions). We define the set $\mathsf{Cart}(A)$ inductively by the following rules:

1. If as is a bad sequence in A^*, then $\mathcal{A}_{as} \in \mathsf{Cart}(A)$ is a name for A_{as}.
2. If $\mathcal{X}_1, \mathcal{X}_2 \in \mathsf{Cart}(A)$ are names for X_1, X_2, then $\mathcal{X}_1 \dot\cup \mathcal{X}_2 \in \mathsf{Cart}(A)$ is a name for $X_1 \dot\cup X_2$.
3. If $\mathcal{X}_1, \mathcal{X}_2 \in \mathsf{Cart}(A)$ are names for X_1, X_2, then $\mathcal{X}_1 \times \mathcal{X}_2 \in \mathsf{Cart}(A)$ is a name for $X_1 \times X_2$.
4. If $\mathcal{X} \in \mathsf{Cart}(A)$ is a name for X, then $\mathcal{X}^* \in \mathsf{Cart}(A)$ is a name for X^*.
5. Let $\mathcal{X}_0, \dots, \mathcal{X}_m \in \mathsf{Cart}(A)$ be names for quasi orders $X_0, \dots, X_m$ and let $0 < \gamma_0 < \dots < \gamma_m \le \omega$. Then $\mathcal{T}\begin{pmatrix}\mathcal{X}_m & \dots & \mathcal{X}_0\\ \gamma_m & \dots & \gamma_0\end{pmatrix}$ is a name for $T\begin{pmatrix}X_m & \dots & X_0\\ \gamma_m & \dots & \gamma_0\end{pmatrix}$.

Remark. Every name $\mathcal{X} \in \mathsf{Cart}(A)$ denotes a set X in an obvious way. Note that $\mathcal{A}_{[]}$ is a name for A. Occasionally we will use the binary operations $\dot\cup$ and $\times$ for finitely many arguments, including the one and zero case. For these cases the definition is to be extended in the obvious way.

Definition (Recursive Definition of $\mathcal{X}^x$ for given $\mathcal{X} \in \mathsf{Cart}(A)$ and $x \in X$).

1. $(\mathcal{A}_{as})^a := \mathcal{A}_{as*a}$.
2. $(\mathcal{X}_1 \dot\cup \mathcal{X}_2)^x := \begin{cases} \mathcal{X}_1{}^x \dot\cup \mathcal{X}_2, & \text{if } x \in X_1,\\ \mathcal{X}_1 \dot\cup \mathcal{X}_2{}^x, & \text{if } x \in X_2.\end{cases}$
3. $(\mathcal{X}_1 \times \mathcal{X}_2)^{(x_1,x_2)} := (\mathcal{X}_1{}^{x_1} \times \mathcal{X}_2) \dot\cup (\mathcal{X}_1 \times \mathcal{X}_2{}^{x_2})$.
4. $(\mathcal{X}^*)^{[x_1,\dots,x_n]} := \dot\bigcup\{(\mathcal{X}^{x_1})^* \times \mathcal{X} \times (\mathcal{X}^{x_2})^* \times \dots \times \mathcal{X} \times (\mathcal{X}^{x_j})^* : j \le n\}$.
5. Let $t \in T\begin{pmatrix}X_m & \dots & X_0\\ \gamma_m & \dots & \gamma_0\end{pmatrix}$ be given. Let t consist of the root $x \in \mathcal{X}_i$ and the subtrees $t_1, \dots, t_n$. We may assume that $\mathcal{T}\begin{pmatrix}\mathcal{X}_m & \dots & \mathcal{X}_0\\ \gamma_m & \dots & \gamma_0\end{pmatrix}^{t_j}$ for all $0 \le j \le n$

is already defined. Set $\mathcal{L} := \mathcal{X}_i \times \dot\bigcup\{(\mathcal{T}(\ldots)^{t_1})^* \times \cdots \times (\mathcal{T}(\ldots)^{t_j})^* : j \leq n\}$. If $n = 0$, i.e. if t only consists of a root, then

$$\mathcal{T}\begin{pmatrix} \mathcal{X}_m & \ldots & \mathcal{X}_0 \\ \gamma_m & \ldots & \gamma_0 \end{pmatrix}^t := \mathcal{T}\begin{pmatrix} \mathcal{X}_m & \ldots & \mathcal{X}_i^x & \ldots & \mathcal{X}_0 \\ \gamma_m & \ldots & \gamma_i & \ldots & \gamma_0 \end{pmatrix}.$$

If $n = \gamma_k$ for $k < i$, we define

$$\mathcal{T}\begin{pmatrix} \mathcal{X}_m & \ldots & \mathcal{X}_0 \\ \gamma_m & \ldots & \gamma_0 \end{pmatrix}^t := \mathcal{T}\begin{pmatrix} \mathcal{X}_m & \ldots & \mathcal{X}_i^x & \ldots & \mathcal{X}_k \dot\cup \mathcal{X}_i \dot\cup \mathcal{L} & \ldots & \mathcal{X}_0 \\ \gamma_m & \ldots & \gamma_i & \ldots & \gamma_k & \ldots & \gamma_0 \end{pmatrix},$$

If $n \neq \gamma_k$ for all $k < i$, then we define

$$\mathcal{T}\begin{pmatrix} \mathcal{X}_m & \ldots & \mathcal{X}_0 \\ \gamma_m & \ldots & \gamma_0 \end{pmatrix}^t := \mathcal{T}\begin{pmatrix} \mathcal{X}_m & \ldots & \mathcal{X}_i^x & \ldots & \mathcal{X}_i \dot\cup \mathcal{L} & \ldots & \mathcal{X}_0 \\ \gamma_m & \ldots & \gamma_i & \ldots & n & \ldots & \gamma_0 \end{pmatrix},$$

where the column with $\mathcal{X}_i \dot\cup \mathcal{L}$ and n has to be filled in at the appropriate place.

Definition. On $\mathsf{Cart}(A)$ we define a relation $<_{\mathsf{Cart}(A)}$ via

$$\mathcal{Y} <_{\mathsf{Cart}(A)} \mathcal{X} \leftrightarrow \exists x \in X\, \mathcal{Y} = \mathcal{X}^x.$$

Lemma 3. Let $\mathcal{X} \in \mathsf{Cart}(A), x \in X$. Then there exists a quasi embedding $e : X_{[x]} \to X^x$.

Proof. $\mathsf{Ind}(\mathsf{Cart}(A))$.

1. Let $a \in A_{as}$. Then, because of $(A_{as})_{[a]} \subseteq (A_{as})^a$, the identity is a quasi embedding from $(A_{as})_{[a]}$ to $(A_{as})^a$.
2. Let $x \in X_1 \dot\cup X_2$. W.l.o.g. $x \in X_1$. By ih there is a quasi embedding $e_{X_1,x} : X_{1[x]} \to {X_1}^x$. We define
$$\begin{aligned} e:\ & (X_1 \dot\cup X_2)_{[x]} \to {X_1}^x \dot\cup X_2 \\ & y \mapsto \begin{cases} e_{X_1,x}(y), & \text{if } y \in X_{1[x]}, \\ y, & \text{if } y \in X_2. \end{cases} \end{aligned}$$
3. Let $(x_1, x_2) \in X_1 \times X_2$. By ih we have quasi embeddings $e_{X_i,x_i} : X_{i[x_i]} \to {X_i}^{x_i}$ for $x_i \in X_i,\ i \in \{1, 2\}$. Set
$$\begin{aligned} e:\ & (X_1 \times X_2)_{[(x_1,x_2)]} \to ({X_1}^{x_1} \times X_2) \dot\cup (X_1 \times {X_2}^{x_2}) \\ & (y_1, y_2) \mapsto \begin{cases} (e_{X_1,x_1}(y_1), y_2), & \text{if } y_1 \in X_{1[x_1]}, \\ (y_1, e_{X_2,x_2}(y_2)), & \text{otherwise.} \end{cases} \end{aligned}$$
4. Let $[x_1, \ldots, x_n] \in X^*$. By ih there are quasi embeddings $e_{X,x_i} : X_{[x_i]} \to X^{x_i}$, for all $1 \leq i \leq n$. We look for a quasi embedding
$$e:\ {X^*}_{[[x_1,\ldots,x_n]]} \to \dot\bigcup\{(X^{x_1})^* \times X \times (X^{x_2})^* \times X \times \cdots \times (X^{x_j})^* : j \leq n\}.$$
If $n = 0$ then the quasi embedding is the empty quasi embedding. Otherwise, let $[y_1, \ldots, y_m] \in {X^*}_{[[x_1,\ldots,x_n]]}$. Because of $[x_1, \ldots, x_n] \not\leq_{X^*} [y_1, \ldots, y_m]$ there exists $0 \leq l < n$ such that $[x_1, \ldots, x_l] \leq_{X^*} [y_1, \ldots, y_m]$ holds, but $[x_1, \ldots, x_{l+1}] \leq_{X^*} [y_1, \ldots, y_m]$ doesn't. Choose $j_1 < \cdots < j_l$ minimal such that $x_i \leq y_{j_i}$, for all $i \leq l$. Then we have $[y_1, \ldots, y_{j_1 - 1}] \in (X_{[x_1]})^*, \ldots, [y_{j_l+1}, \ldots, y_m] \in (X_{[x_{l+1}]})^*$, and by induction hypothesis we may define $ws_1 := [e_{X,x_1}(y_1), \ldots, e_{X,x_1}(y_{j_1-1})] \in (X^{x_1})^*, \ldots,$ and $ws_{l+1} := [e_{X,x_{l+1}}(y_{j_l+1}), \ldots, e_{X,x_{l+1}}(y_m)] \in (X^{x_{l+1}})^*$. We set
$$\begin{aligned} & e([y_1, \ldots, y_m]) := \\ & (ws_1, y_{j_1}, ws_2, y_{j_2}, \ldots, y_{j_l}, ws_{l+1}). \end{aligned}$$

5. Let $t \in T(\begin{smallmatrix} X_m & \dots & X_0 \\ \gamma_m & \dots & \gamma_0 \end{smallmatrix})$ be given. Ind(structure of t). Let us regard first the case t consisting only of a root with a label $x \in X_i$. Let $u \in T(\begin{smallmatrix} X_m & \dots & X_0 \\ \gamma_m & \dots & \gamma_0 \end{smallmatrix})_{[t]}$. Then for every node in u with a label $y \in X_i$ it holds $x \not\leq_{X_i} y$, i.e. u lies in $T\binom{X_m \ \dots \ X_{i[x]} \ \dots \ X_0}{\gamma_m \ \dots \ \gamma_i \ \dots \ \gamma_0}$. A quasi embedding into $T\binom{X_m \ \dots \ X_i{}^x \ \dots \ X_0}{\gamma_m \ \dots \ \gamma_i \ \dots \ \gamma_0}$ can easily be constructed by ih, applied to to X_i and x.

Now, assume t consists of a root $x \in X_i$ and the immediate sub trees $t_1, \dots, t_n$ with $n = \gamma_k < \gamma_i$ (The case $n \neq \gamma_k$, for all $k < i$ runs analogously). By induction hypothesis, for all $j \leq n$ there exist quasi embeddings

$$e_{t_j} : T\binom{X_m \ \dots \ X_0}{\gamma_m \ \dots \ \gamma_0}_{[t_j]} \to T\binom{X_m \ \dots \ X_0}{\gamma_m \ \dots \ \gamma_0}^{t_j}.$$

We look for a quasi embedding

$$e : T\binom{X_m \ \dots \ X_0}{\gamma_m \ \dots \ \gamma_0}_{[t]} \to T\binom{X_m \ \dots \ X_i{}^x \ \dots \ X_k \dot\cup X_i \dot\cup L \ \dots \ X_0}{\gamma_m \ \dots \ \gamma_i \ \dots \ \gamma_k \ \dots \ \gamma_0}.$$

Let $u \in T\binom{X_m \ \dots \ X_0}{\gamma_m \ \dots \ \gamma_0}_{[t]}$ be a tree with root y and immediate subtrees $u_1, \dots, u_r$. We may assume that the quasi embeddings on the subtrees are already defined, and have to map the root label y suitably to a label in $X_0, \dots, X_k \dot\cup X_i \dot\cup L, \dots, X_i{}^x, \dots,$ or X_m such that the condition concerning the number of immediate subtrees is fulfilled.

$t \not\leq_{T(\dots)} u$ only can hold for one of the following reasons:

i. $y \notin X_i$. Then map y to itself respectively in the case $y \in X_k$ to y in the first component of $X_k \dot\cup X_i \dot\cup L$. We set $e(y\, u_1 \dots u_r) := y\, e(u_1) \dots e(u_r)$.

ii. $y \in X_i$, but $x \not\leq y$. Hence $y \in X_{i[x]}$ and by induction hypothesis we have $e_{X_i,x}(y) \in X_i{}^x$. We set $e(y\, u_1 \dots u_r) := e_{X_i,x}(y)\, e(u_1) \dots e(u_r)$.

iii. $y \in X_i$ and $x \leq y$, but y has less than γ_k successors. Then anyhow we may map y to itself, if we regard y as a label in the second component of $X_k \dot\cup X_i \dot\cup L$. We set $e(y\, u_1 \dots u_r) := y\, e(u_1) \dots e(u_r)$.

iv. $y \in X_i$, $x \leq y$ and y has more than γ_k successors, but $[t_1, \dots, t_{\gamma_k}] \not\leq_{T(\dots)^*} [u_1, \dots, u_r]$. Now then there exists l, $0 \leq l < \gamma_k$ such that $[t_1, \dots, t_l] \leq_{T(\dots)^*} [u_1, \dots, u_r]$ and $[t_1, \dots, t_{l+1}] \not\leq_{T(\dots)^*} [u_1, \dots, u_r]$. We will be mapping u to a tree with $l < \gamma_k$ subtrees and a root label in L, where this label contains the images of the remaining subtrees. More formally: Choose $j_1 < \dots < j_l$ minimal such that $t_1 \leq_{T(\dots)} u_{j_1}, \dots, t_l \leq_{T(\dots)} u_{j_l}$. Then it holds

$$[u_1, \dots, u_{j_1 - 1}] \in T\binom{X_m \ \dots \ X_0}{\gamma_m \ \dots \ \gamma_0}_{[t_1]}^*,$$
$$\dots$$
$$[u_{j_l+1}, \dots, u_r] \in T\binom{X_m \ \dots \ X_0}{\gamma_m \ \dots \ \gamma_0}_{[t_{l+1}]}^*.$$

and by ih it follows

$$ts_1 := [e_{t_1}(u_1), \dots, e_{t_1}(u_{j_1-1})] \in T\binom{X_m \ \dots \ X_0}{\gamma_m \ \dots \ \gamma_0}^{t_1 *}$$
$$\dots$$
$$ts_{l+1} := [e_{t_{l+1}}(u_{j_l+1}), \dots, e_{t_{l+1}}(u_r)] \in T\binom{X_m \ \dots \ X_0}{\gamma_m \ \dots \ \gamma_0}^{t_{l+1} *}.$$

Finally we put

$$e(y\, u_1 \dots u_r) := (\, y,\, (ts_1, \dots, ts_{l+1}))\, e(u_{j_1}) \dots e(u_{j_l}).$$

It is left to the reader to check that e is actually a quasi embedding. $\diamond$

Lemma 4. $\forall \mathcal{X} \in \mathsf{Cart}(A).\, \mathsf{acc}_{<_{\mathsf{Cart}(A)}} \mathcal{X} \to \mathsf{acc}_{\ll_X} []$.

Proof. $\mathsf{Ind}(\mathsf{acc}_{<_{\mathsf{Cart}(A)}})$. Let $\mathcal{X} \in \mathsf{Cart}(A)$. Assume $\mathsf{ih} : \forall \mathcal{Y} <_{\mathsf{Cart}(A)} \mathcal{X}\, \mathsf{acc}_{\ll_Y} []$. Because of lemma 1 it suffices to show $\forall x \in X\, \mathsf{acc}_{\ll_{X_{[x]}}} []$. Let $x \in X$. By lemma 3 there exists a quasi embedding $e : X_{[x]} \to X^x$. Therefore, using lemma 2 we only have to show $\mathsf{acc}_{\ll_{X^x}} []$. But this follows by ih. $\diamond$

Definition. On the set

$$\mathsf{Lex} := \left\{ \begin{pmatrix} \mathcal{X}_n & \dots & \mathcal{X}_1 \\ \gamma_n & \dots & \gamma_1 \end{pmatrix},\ \mathsf{acc}_{<_{\mathsf{Cart}(A)}} \mathcal{X}_i,\ i \le n,\ 0 \le n < \omega,\ 0 < \gamma_1 < \dots < \gamma_n \le \omega \right\}$$

we define a relation $<_{\mathsf{Lex}}$: $\begin{pmatrix} \mathcal{Y}_m & \dots & \mathcal{Y}_1 \\ \delta_m & \dots & \delta_1 \end{pmatrix} <_{\mathsf{Lex}} \begin{pmatrix} \mathcal{X}_n & \dots & \mathcal{X}_1 \\ \gamma_n & \dots & \gamma_1 \end{pmatrix} \iff$

$$\begin{aligned} & m, n > 0 \text{ and} \\ & \delta_m = \gamma_n \ \wedge\ \mathcal{Y}_m <_{\mathsf{Cart}(A)} \mathcal{X}_n \text{ or} \\ & \delta_m = \gamma_n \ \wedge\ \mathcal{Y}_m = \mathcal{X}_n \ \wedge\ \begin{pmatrix} \mathcal{Y}_{m-1} & \dots & \mathcal{Y}_1 \\ \delta_{m-1} & \dots & \delta_1 \end{pmatrix} <_{\mathsf{Lex}} \begin{pmatrix} \mathcal{X}_{n-1} & \dots & \mathcal{X}_1 \\ \gamma_{n-1} & \dots & \gamma_1 \end{pmatrix}. \end{aligned}$$

Lemma 5. Assume $\mathsf{acc}_{<_{\mathsf{Cart}(A)}} \mathcal{X}_0, \dots, \mathsf{acc}_{<_{\mathsf{Cart}(A)}} \mathcal{X}_n$ and let $0 < \gamma_0 < \dots < \gamma_n \le \omega$. Then it follows $\mathsf{acc}_{<_{\mathsf{Lex}}} \begin{pmatrix} \mathcal{X}_n & \dots & \mathcal{X}_0 \\ \gamma_n & \dots & \gamma_0 \end{pmatrix}$.

Proof. We show

$$\begin{aligned} & \forall \gamma \le \omega.\, \forall \mathcal{X}.\, \mathsf{acc}_{<_{\mathsf{Cart}(A)}} \mathcal{X} \to \forall \mathbb{X}.\, \mathsf{acc}_{<_{\mathsf{Lex}}} \mathbb{X} \to \\ & \forall n, \mathcal{X}_1, \dots, \mathcal{X}_n, \gamma_1, \dots, \gamma_n.\, \gamma_1 < \dots < \gamma_n < \gamma \wedge \mathbb{X} = \begin{pmatrix} \mathcal{X}_n & \dots & \mathcal{X}_1 \\ \gamma_n & \dots & \gamma_1 \end{pmatrix} \to \\ & \mathsf{acc}_{<_{\mathsf{Lex}}} \begin{pmatrix} \mathcal{X} & \mathcal{X}_n & \dots & \mathcal{X}_1 \\ \gamma & \gamma_n & \dots & \gamma_1 \end{pmatrix} \end{aligned}$$

by $\mathsf{Ind}_1(\gamma)$, $\mathsf{Ind}_2(\mathsf{acc}_{<_{\mathsf{Cart}(A)}})$ and $\mathsf{Ind}_3(\mathsf{acc}_{<_{\mathsf{Lex}}})$. We need to prove $\mathsf{acc}_{<_{\mathsf{Lex}}} \mathbb{Y}$ for all $\mathbb{Y} <_{\mathsf{Lex}} \begin{pmatrix} \mathcal{X} & \mathcal{X}_n & \dots & \mathcal{X}_1 \\ \gamma & \gamma_n & \dots & \gamma_1 \end{pmatrix}$

i. Let $\mathbb{Y} = \begin{pmatrix} \mathcal{X}^x & \mathcal{Y}_m & \dots & \mathcal{Y}_1 \\ \gamma & \delta_m & \dots & \delta_1 \end{pmatrix}$ such that $x \in X$, $\delta_1 < \dots < \delta_m < \gamma$ and $\forall i \le m\, \mathsf{acc}_{<_{\mathsf{Cart}(A)}} \mathcal{Y}_i$. By an m-fold application of ih_1 we obtain $\mathsf{acc}_{<_{\mathsf{Lex}}} \begin{pmatrix} \mathcal{Y}_m & \dots & \mathcal{Y}_1 \\ \delta_k & \dots & \delta_1 \end{pmatrix}$ and by ih_2 we may conclude $\mathsf{acc}_{<_{\mathsf{Lex}}} \begin{pmatrix} \mathcal{X}^x & \mathcal{Y}_m & \dots & \mathcal{Y}_1 \\ \gamma & \delta_m & \dots & \delta_1 \end{pmatrix}$.

ii. Let $\mathbb{Y} = \begin{pmatrix} \mathcal{X} & \mathcal{Y}_m & \dots & \mathcal{Y}_1 \\ \gamma & \delta_m & \dots & \delta_1 \end{pmatrix}$ such that $\begin{pmatrix} \mathcal{Y}_m & \dots & \mathcal{Y}_1 \\ \delta_m & \dots & \delta_1 \end{pmatrix} <_{\mathsf{Lex}} \begin{pmatrix} \mathcal{X}_n & \dots & \mathcal{X}_1 \\ \gamma_n & \dots & \gamma_1 \end{pmatrix}$. Then $\mathsf{acc}_{<_{\mathsf{Lex}}} \begin{pmatrix} \mathcal{X} & \mathcal{Y}_m & \dots & \mathcal{Y}_1 \\ \gamma & \delta_m & \dots & \delta_1 \end{pmatrix}$ immediately follows by ih_3.

The lemma follows from $\mathsf{acc}_{<_{\mathsf{Lex}}} ()$ by $n+1$ applications of this assertion. $\diamond$

Lemma 6. Assume $\mathsf{acc}_{\ll_A}[]$. Then $\forall \mathcal{X}. \mathcal{X} \in \mathsf{Cart}(A) \to \mathsf{acc}_{<_{\mathsf{Cart}(A)}} \mathcal{X}$.

Proof. $\mathsf{Ind}(\mathsf{Cart}(A))$.

1. We show $\forall as \in \mathrm{Bad}(A). \mathsf{acc}_{\ll_A} as \to \mathsf{acc}_{<_{\mathsf{Cart}(A)}} \mathcal{A}_{as}$. $\mathsf{Ind}(\mathsf{acc}_{\ll_A})$. Let $as \in \mathrm{Bad}(A)$ and assume ih: $\forall bs \ll_A as\, \mathsf{acc}_{<_{\mathsf{Cart}(A)}} \mathcal{A}_{bs}$. We need to prove $\forall a \in A_{as}\, \mathsf{acc}_{<_{\mathsf{Cart}(A)}} (\mathcal{A}_{as})^a$. But this follows by ih because of $(\mathcal{A}_{as})^a = \mathcal{A}_{as*a}$ and $as * a \ll_A as$.
2. Let $\mathcal{X}_1$ and $\mathcal{X}_2$ s.t. $\mathsf{acc}_{<_{\mathsf{Cart}(A)}} \mathcal{X}_1$ and $\mathsf{acc}_{<_{\mathsf{Cart}(A)}} \mathcal{X}_2$. In order to show $\mathsf{acc}_{<_{\mathsf{Cart}(A)}} \mathcal{X}_1 \dot\cup \mathcal{X}_2$ we use induction on $\mathsf{acc}_{<_{\mathsf{Cart}(A)}} \mathcal{X}_1$ and $\mathsf{acc}_{<_{\mathsf{Cart}(A)}} \mathcal{X}_2$. Let $x \in \mathcal{X}_1 \dot\cup \mathcal{X}_2$, w.l.o.g. $x \in X_1$, then the induction hypothesis implies $\mathsf{acc}_{<_{\mathsf{Cart}(A)}} {\mathcal{X}_1}^{x_1} \dot\cup \mathcal{X}_2$.
3. Let $\mathcal{X}_1$ and $\mathcal{X}_2$ s.t. $\mathsf{acc}_{<_{\mathsf{Cart}(A)}} \mathcal{X}_1$ and $\mathsf{acc}_{<_{\mathsf{Cart}(A)}} \mathcal{X}_2$. We show $\mathsf{acc}_{<_{\mathsf{Cart}(A)}} \mathcal{X}_1 \times \mathcal{X}_2$ by induction on $\mathsf{acc}_{<_{\mathsf{Cart}(A)}} \mathcal{X}_1$ and $\mathsf{acc}_{<_{\mathsf{Cart}(A)}} \mathcal{X}_2$. We have

   $\mathsf{ih}_1 : \forall x_1 \in X_1, \forall \mathcal{X}_2. \mathsf{acc}_{<_{\mathsf{Cart}(A)}} \mathcal{X}_2 \to \mathsf{acc}_{<_{\mathsf{Cart}(A)}} \mathcal{X}_1^{x_1} \times \mathcal{X}_2$

   $\mathsf{ih}_2 : \forall x_2 \in X_2\, \mathsf{acc}_{<_{\mathsf{Cart}(A)}} \mathcal{X}_1 \times \mathcal{X}_2^{x_2}$

   Now, let $(x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2$. $\mathsf{acc}_{<_{\mathsf{Cart}(A)}} (\mathcal{X}_1^{x_1} \times \mathcal{X}_2) \dot\cup (\mathcal{X}_1 \times \mathcal{X}_2^{x_2})$ follows by ih_1, applied to x_1 and $\mathcal{X}_2$, ih_2 and 2.
4. Let $\mathcal{X}$ such that $\mathsf{acc}_{<_{\mathsf{Cart}(A)}} \mathcal{X}$ be given. We prove $\mathsf{acc}_{<_{\mathsf{Cart}(A)}} \mathcal{X}^*$ by induction on $\mathsf{acc}_{<_{\mathsf{Cart}(A)}} \mathcal{X}$. Assume $\mathsf{ih} : \forall x \in \mathcal{X}\, \mathsf{acc}_{<_{\mathsf{Cart}(A)}} (\mathcal{X}^x)^*$. We have to show $\mathsf{acc}_{<_{\mathsf{Cart}(A)}} \mathcal{X}^*$, i.e. $\forall w \in X^*\, \mathsf{acc}_{<_{\mathsf{Cart}(A)}} (\mathcal{X}^*)^w$. Let $w = [x_1, \ldots, x_n]$. Then $\mathsf{acc}_{<_{\mathsf{Cart}(A)}} \dot\bigcup\{(\mathcal{X}^{x_1})^* \times \mathcal{X} \times \cdots \times \mathcal{X} \times (\mathcal{X}^{x_j})^* : j \le n\}$ follows by ih, 2., and 3.
5. Let $\mathcal{X}_0, \ldots, \mathcal{X}_m$ such that $\mathsf{acc}_{<_{\mathsf{Cart}(A)}} \mathcal{X}_i$ for all $i \le m$ and $0 < \gamma_0 < \cdots < \gamma_m \le \omega$. Since $\mathsf{acc}_{<_{\mathsf{Cart}(A)}} \mathcal{X}_0, \ldots, \mathsf{acc}_{<_{\mathsf{Cart}(A)}} \mathcal{X}_m$ implies $\mathsf{acc}_{<_{\mathsf{Lex}}} \binom{\mathcal{X}_m \ \cdots \ \mathcal{X}_0}{\gamma_m \ \cdots \ \gamma_0}$ by lemma 5, we obtain $\mathsf{acc}_{<_{\mathsf{Cart}(A)}} \mathcal{T}\binom{\mathcal{X}_m \ \cdots \ \mathcal{X}_0}{\gamma_m \ \cdots \ \gamma_0}$ when we have shown more generally

   $$\forall \mathbb{X} \neq (). \mathsf{acc}_{<_{\mathsf{Lex}}} \mathbb{X} \to \mathsf{acc}_{<_{\mathsf{Cart}(A)}} \mathcal{T}(\mathbb{X}).$$

   $\mathsf{Ind}_1(\mathsf{acc}_{<_{\mathsf{Lex}}})$. Fix $\mathbb{X}$ and assume $\mathsf{ih}_1 : \forall \mathbb{Y} <_{\mathsf{Lex}} \mathbb{X}. \mathsf{acc}_{<_{\mathsf{Cart}(A)}} \mathcal{T}(\mathbb{Y})$. Suppose $\mathbb{X}$ to be of a form $\binom{\mathcal{X}_r \ \cdots \ \mathcal{X}_0}{\gamma_r \ \cdots \ \gamma_0}$ show $\mathsf{acc}_{<_{\mathsf{Cart}(A)}} \mathcal{T}(\mathbb{X})^t$ for an arbitrary $t \in T(\mathbb{X})$ by Ind_2 (structure of t). Assume first that t is a branch with a label $x \in X_i$. Since $\binom{\mathcal{X}_r \ \cdots \ \mathcal{X}_i^x \ \cdots \ \mathcal{X}_0}{\gamma_r \ \cdots \ \gamma_i \ \cdots \ \gamma_0} <_{\mathsf{Lex}} \mathbb{X}$ we may conclude $\mathsf{acc}_{<_{\mathsf{Cart}(A)}} \mathcal{T}\binom{\mathcal{X}_r \ \cdots \ \mathcal{X}_i^x \ \cdots \ \mathcal{X}_0}{\gamma_r \ \cdots \ \gamma_i \ \cdots \ \gamma_0}$ by ih_1.

   Now, assume that t consists of the root $x \in \mathcal{X}_i$ and the subtrees $t_1, \ldots, t_n$, $n = \gamma_k < \gamma_i$. (The case $n \neq \gamma_k$ for all $k < i$ is analogous.) By ih_2 it follows $\mathsf{acc}_{<_{\mathsf{Cart}(A)}} \mathcal{T}\binom{\mathcal{X}_r \ \cdots \ \mathcal{X}_0}{\gamma_r \ \cdots \ \gamma_0}^{t_j}$, for all $j \le n$. Because of 2., 3., and 4. we are able to conclude $\mathsf{acc}_{<_{\mathsf{Cart}(A)}} (\mathcal{X}_i \times \dot\bigcup\{(\mathcal{T}(\mathbb{X})^{t_1})^* \times \cdots \times (\mathcal{T}(\mathbb{X})^{t_j})^* : j \le n\})$. By using the abbreviation $\mathcal{L} = \mathcal{X}_i \times \dot\bigcup\{(\mathcal{T}(\mathbb{X})^{t_1})^* \times \cdots \times (\mathcal{T}(\mathbb{X})^{t_j})^* : j \le n\}$ we get $\mathsf{acc}_{<_{\mathsf{Cart}(A)}} \mathcal{L}$ and $\mathsf{acc}_{<_{\mathsf{Cart}(A)}} (\mathcal{X}_k \dot\cup \mathcal{X}_i \dot\cup \mathcal{L})$ by a further application of 2. Therefore $\mathcal{T}\binom{\mathcal{X}_r \ \cdots \ \mathcal{X}_i^x \ \cdots \ \mathcal{X}_k \dot\cup \mathcal{X}_i \dot\cup \mathcal{L} \ \cdots \ \mathcal{X}_0}{\gamma_r \ \cdots \ \gamma_i \ \cdots \ \gamma_k \ \cdots \ \gamma_0} <_{\mathsf{Lex}} \mathcal{T}\binom{\mathcal{X}_r \ \cdots \ \mathcal{X}_i \ \cdots \ \mathcal{X}_k \ \cdots \ \mathcal{X}_0}{\gamma_r \ \cdots \ \gamma_i \ \cdots \ \gamma_k \ \cdots \ \gamma_0}$ and we may apply ih_1 and obtain $\mathsf{acc}_{<_{\mathsf{Cart}(A)}} \mathcal{T}\binom{\mathcal{X}_r \ \cdots \ \mathcal{X}_i^x \ \cdots \ \mathcal{X}_k \dot\cup \mathcal{X}_i \dot\cup \mathcal{L} \ \cdots \ \mathcal{X}_0}{\gamma_r \ \cdots \ \gamma_i \ \cdots \ \gamma_k \ \cdots \ \gamma_0}$.

◇

Proof of Kruskal's theorem. We want to show $\mathsf{acc}_{\ll_A}[] \to \mathsf{acc}_{\ll_{T\binom{A}{\omega}}}[]$. Assume $\mathsf{acc}_{\ll_A}[]$. For all $\mathcal{X} \in \mathsf{Cart}(A)$ we have $\mathsf{acc}_{<_{\mathsf{Cart}(A)}} \mathcal{X}$ by lemma 6 and therefore

$\mathsf{acc}_{\ll_X}$ [] by lemma 4. This holds especially for $\mathcal{X} = \mathcal{T}(\binom{\mathcal{A}_{[]}}{\omega})$ and since $A_{[]} = A$ we may conclude $\mathsf{acc}_{\ll_{\mathcal{T}(\binom{A}{\omega})}}$ []. $\diamond$

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