Stability of representations of effective partial algebras

Jens Blanck∗1, Viggo Stoltenberg-Hansen2, and John V. Tucker1

1 Department of Computer Science, Swansea University, Singleton Park, Swansea, SA2 8PP, Wales
2 Department of Mathematics, Uppsala University, Box 480, SE-751 06 Uppsala, Sweden

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An algebra is effective if its operations are computable under some numbering. When are two numberings of an effective partial algebra equivalent? For example, the computable real numbers form an effective field and two effective numberings of the field of computable reals are equivalent if the limit operator is assumed to be computable in the numberings (theorems of Moschovakis and Hertling). To answer the question for effective algebras in general, we give a general method based on an algebraic analysis of approximations by elements of a finitely generated subalgebra. Commonly, the computable elements of a topological partial algebra are derived from such a finitely generated algebra and form a countable effective partial algebra. We apply the general results about partial algebras to the recursive reals, ultrametric algebras constructed by inverse limits, and to metric algebras in general.

1 Introduction

An effective algebra A is an algebra whose operations are tracked or simulated by functions computable with respect to some effective representation R of the algebra A. The standard method of building representations of algebras is to use numberings, i.e., surjections from a subset of the natural numbers N to the underlying carrier set of the algebra A. The computability of the algebra A is defined by the computability of the numerical representation. An effective algebra requires only the operations of A to be computable on the numerical codes; neither the code set nor the algebra’s relations, such as equality, need be computable. The operations may be partial; for example, the operation of inverse in a field is usually (but not always1) a partial operation.

An important example that motivates our interest in effective partial algebras is the field of computable real numbers. The construction of the effective field of computable real numbers from the computable field of rationals is an instance of a quite general phenomenon. An algebra A is created as the “completion” of an algebra D with respect to some notion or purpose. Typically, A is intended to encompass the scope of some notion of “approximation by the elements of D”; furthermore, the notion of approximation induces a topology so that the algebra A and the subalgebra D become topological algebras with D dense in A. The question arises: What part of the completion A of D is algorithmic?

Suppose there exists an effective subalgebra D of the algebra A and some notion of approximation. We can now consider the subset AD,k of A consisting of those elements of A that can be effectively approximated by a sequence in D. The set AD,k contains the computable elements of A, as determined by the effective subalgebra D. Ideally, AD,k is a subalgebra of A and is itself an effective algebra. For example, the effective field of computable real numbers is the subfield RQ,k made by the effective completion of the effective – indeed, computable – rational field Q. To what extent does computability on AD,k depend upon the choice of numberings of the subalgebra D? How stable or invariant is the derived class of computable functions? Constructions such as $D \preceq A_{D,k} \preceq A$ constitute the foundations of computable analysis.

∗ Corresponding author: e-mail: j.e.blanck@swansea.ac.uk

† The partiality of operations can be a subtle matter. In topological algebra, operations are made partial to avoid discontinuities. Inverse on R is discontinuous at 0 if one defines $0^{-1} = 0$, say. However, there is a perfectly workable algebraic theory of “fields with total division” called meadows: see Bergstra et al. [1].
In this paper we consider effective partial algebras in general; and we ask and answer the following general algebraic question:

**When are two numberings of an effective partial algebra equivalent?**

We give a general method for showing that all effective numberings of certain partial algebras are recursively equivalent. The method is based on an algebraic analysis of “approximating” elements of \( A \) using a subalgebra \( D \). Furthermore, we focus on using finitely generated subalgebras. When Mal’cev [2] launched the theory of numberings, he showed that finitely generated algebras are computably stable, i.e., all computable numberings are equivalent, see also [7, 8]. We extend this result to the completions of the finitely generated subalgebra \( D \).

Moschovakis [3] and Hertling [4], have shown that two numberings of the computable reals are equivalent if the limit operator is assumed to be computable in the numberings. Reflecting on these theorems, the process of obtaining a sequence of approximations of elements is identified as the important step in the existing proofs for the reals. We formulate the concept of approximation-limit pairs, where the approximation and the limit processes are linked formally in an algebraic way. In general, approximation is a relation between elements of the algebra and elements of a finitely generated subalgebra, with a natural number specifying the level or degree of the approximation. An approximation-limit pair is an algebraic abstraction akin to normal forms. There are many subtle properties at the level of the representations, however. We would like to computably select approximations from a code of an element, but this is not a well-defined function on the level of the algebra, as the selection often depends on the code rather than the element it denotes. We use a weaker notion of computable (non-deterministic) selection.

The main result (Theorem 4.6) dissects the intricate dependencies between numberings; it has this corollary:

**Theorem** An algebra with an approximation-limit pair which has computable selection and a computable limit process has at most one numbering up to recursive equivalence.

We also prove an existence result (Theorem 4.10) giving sufficient conditions (e.g., effective continuity) for the existence of an effective numbering for certain algebras of computable elements.

We apply the general results about partial algebras to the recursive reals, ultrametric algebras constructed by inverse limits, and to metric algebras in general.

## 2 Computability and numberings

Extensive background on computability on numbered structures can be found in Mal’cev [2], Ershov [5, 6], and Stoltenberg-Hansen and Tucker [7, 8].

We use recursion theory as the underlying computability theory on the natural numbers. We assume very basic knowledge of recursion theory as can be found in any basic text. Our terminology and notation is standard. In particular, we let \((\varphi_e)_{e \in \mathbb{N}}\) be a standard numbering of the partial recursive functions, and \((W_e)_{e \in \mathbb{N}}\) the corresponding numbering of the recursively enumerable (r.e.) sets. We use \(\downarrow\) and \(\uparrow\) to denote convergence and divergence of a computation respectively. The strong (Kleene) equality is denoted by \(\simeq\).

### 2.1 Numberings and partial functions

Let \( A \) be a set. A **numbering** of \( A \) is a surjective function \( \alpha : \Omega_\alpha \to A \), where \( \Omega_\alpha \subseteq \mathbb{N} \). It should be thought of as a coding of \( A \) by natural numbers. A **numbered set** is a pair \((A, \alpha)\) such that \( \alpha \) is a numbering of \( A \). The kernel of \( \alpha \) is \( \{\alpha^{-1}(n) : n \in \mathbb{N}\} \). If \((A, \alpha)\) and \((B, \beta)\) are numbered sets then \( \alpha \times \beta : \Omega_{\alpha \times \beta} \to A \times B \) is the numbering \( \alpha \times \beta((m, n)) = (\alpha(m), \beta(n)) \), where \( \Omega_{\alpha \times \beta} = \{(m, n) : m \in \Omega_\alpha, n \in \Omega_\beta\} \) and \((\cdot, \cdot)\) is a standard recursive pairing function.

A subset \( S \) of a numbered set \((A, \alpha)\) is **\( \alpha \)-semicomputable** if there exists an r.e. set \( W \) such that \( \alpha^{-1}[S] = \Omega_\alpha \cap W \).

**Definition 2.1** Let \( \alpha \) and \( \beta \) be numberings of a set \( A \).

(i) \( \alpha \) recursively reduces to \( \beta \), denoted \( \alpha \leq \beta \), if there is a partial recursive function \( f \) such that for each \( n \in \Omega_\alpha \), \( \alpha(n) \simeq \beta(f(n)) \).

(ii) \( \alpha \) is recursively equivalent to \( \beta \), denoted \( \alpha \sim \beta \), if \( \alpha \leq \beta \) and \( \beta \leq \alpha \).

For convenience, we repeat the definitions of computability for partial functions from [9].
Definition 2.2 Let \((A, \alpha)\) and \((B, \beta)\) be numbered sets and let \(f : A \rightarrow B\) be a partial function.

\[
\begin{array}{c|c}
A & f \\
\alpha & \beta \\
\hline
\Omega_\alpha & \bar{f} \\
\Omega_\beta & 
\end{array}
\]

(i) \(f\) is \((\alpha, \beta)\)-computable if there exists a partial recursive function \(\bar{f} : \mathbb{N} \rightarrow \mathbb{N}\) such that for each \(n \in \Omega_\alpha\),

(a) \(\bar{f}(n) \downarrow \Rightarrow \bar{f}(n) \in \Omega_\beta\); and

(b) \(f(\alpha(n)) \simeq \beta(\bar{f}(n))\).

(ii) \(f\) is weakly \((\alpha, \beta)\)-computable if there exists a partial recursive function \(\bar{f} : \mathbb{N} \rightarrow \mathbb{N}\) such that

\[
f(\alpha(n)) \downarrow \Rightarrow f(\alpha(n)) \simeq \beta(\bar{f}(n)).
\]

In either case, we say that \(\bar{f}\) tracks \(f\).

The following is a useful observation.

Lemma 2.3 Let \((A, \alpha)\) and \((B, \beta)\) be numbered sets and let \(f : A \rightarrow B\) be a partial function. Then \(f\) is \((\alpha, \beta)\)-computable if, and only if, \(f\) is weakly \((\alpha, \beta)\)-computable and \(\text{dom } f\) is \(\alpha\)-semicomputable.

Thus, for total functions computable and weakly computable coincide. An example of an operation that is weakly computable, but not, in general, computable, is the limit operator on Cauchy sequences. This is because it is impossible to check the Cauchy criterion for the full sequence, i.e., the domain of the tracking function will include non-Cauchy sequences. On the other hand, inverting a real number is a computable partial function since the set of non-zero numbers is semicomputable.

The standard proofs of the following for partial recursive functions lift easily to numbered sets.

Proposition 2.4 Let \((A, \alpha)\) and \((B, \beta)\) be numbered sets, let \(R_1, \ldots, R_n \subseteq A\) be disjoint \(\alpha\)-semicomputable sets, and let \(g_1, \ldots, g_n\) be \((weakly) \ (\alpha, \beta)\)-computable partial functions \(g_i : A \rightarrow B\). Define the partial function \(f : A \rightarrow B\) by

\[
f(x) \simeq \begin{cases} 
g_1(x), & \text{if } R_1(x); \\
\vdots \\
g_n(x), & \text{if } R_n(x). 
\end{cases}
\]

Then \(f\) is \((weakly) \ (\alpha, \beta)\)-computable.

Corollary 2.5 Let \((A, \alpha)\) be a numbered set. Then \(R\) and \(A \setminus R\) are both \(\alpha\)-semicomputable if, and only if, the characteristic function \(\chi_R : A \rightarrow \{0, 1\}\) is \(\alpha\)-computable.

2.2 Computable non-deterministic selection

We say that a relation \(R \subseteq A \times B\) is left-total if each \(x \in A\) is related to some \(y \in B\), or equivalently, if the projection to the first coordinate is surjective. For a left-total relation we would like a function \(s : A \rightarrow B\), called a selection function, such that for each \(x \in A, x R s(x)\). However, in our situation it is not always possible to effectively track such a selection function for numbered sets because the selection may be dependent on the representation of \(x\) rather than \(x\) itself. These considerations lead to the following weaker definition of computable selection.
Definition 2.6 Let \( (A, \alpha) \) and \( (B, \beta) \) be numbered sets, and let \( R \subseteq A \times B \) be a left-total relation. The relation \( R \) has \( (\alpha, \beta) \)-computable (non-deterministic) selection if there exists a partial recursive function \( f \) s.t. \( f(\Omega_\alpha) \subseteq \Omega_\beta \), and for all \( n \in \Omega_\alpha \)

(i) \( f(n) \downarrow \), and
(ii) \( (\alpha(n), \beta(f(n))) \in R \).

We say that \( f \) tracks the selection for \( R \).

Note that there is no requirement that \( m \equiv_\alpha n \) implies \( f(m) \equiv_\beta f(n) \), hence \( f \) need not induce a well-defined function from \( A \) to \( B \). On the level of the sets \( A \) and \( B \) the selection is non-deterministic. Other authors have chosen to model this behaviour as multi-valued functions, e.g., Weihrauch [10].

Proposition 2.7 Let \( (A, \alpha) \) and \( (B, \beta) \) be numbered sets, where \( \Omega_\beta \) is r.e. Then each \( \alpha \times \beta \)-semicomputable left-total relation \( R \subseteq A \times B \) has \( (\alpha, \beta) \)-computable selection.

Proof. Let \( W \) be an r.e. set witnessing that \( R \) is semicomputable. The selection is tracked by
\[
f(n) = \nu m((n, m) \in W \land \beta(m) \downarrow).
\]

3 Effective partial \( \Sigma \)-algebras

In this section we review some of the results in [9].

Definition 3.1 Let \( A \) be a partial \( \Sigma \)-algebra and let \( \alpha \) be a numbering of \( A \). Then \( (A, \alpha) \) is a (weakly) effective partial \( \Sigma \)-algebra, and \( \alpha \) is an effective numbering, if each \( k \)-ary partial operation \( \sigma \) of \( A \) is (weakly) \( (\alpha^k, \alpha) \)-computable, where \( \alpha^k \) is the product numbering of \( A^k \) obtained from \( \alpha \).

The ordered field of recursive numbers with the standard numbering is an effective partial \( \Sigma \)-algebra.

In the sequel we assume that there is a computable enumeration of the operation symbols in \( \Sigma \) along with their arities. In particular this is true when \( \Sigma \) is finite. Then the total term algebra \( T(\Sigma, V) \), where \( V = \{v_0, v_1, v_2, \ldots\} \) is a countable set of variables, has a standard numbering which we denote by \( \gamma \) (see [7]).

Let \( A \) be a partial \( \Sigma \)-algebra and let \( \nu: \mathbb{N} \to A \) be a partial sequence in \( A \). Then we let \( \text{TE}_e: T(\Sigma, V) \to A \) be the corresponding partial term evaluation map (sending \( v_i \) to \( e(i) \)). Define the partial function \( \gamma_e: \mathbb{N} \to A \) by
\[
\gamma_e(n) \simeq \text{TE}_e(\gamma(n)).
\]

We denote \( \text{dom}(\gamma_e) \) by \( \Omega_e \).

Let \( (e) \) be the partial \( \Sigma \)-subalgebra generated by the partial sequence \( e \), i.e., \( (e) \) is the image of \( \gamma_e \). It is shown in [9] that \( (e), \gamma_e \) is a weakly effective partial \( \Sigma \)-algebra.

We say that a partial sequence \( e: \mathbb{N} \to A \), where \( A \) is a set numbered by \( \alpha \), is (weakly) \( \alpha \)-computable if it is (weakly) \( (\text{id}, \alpha) \)-computable. Note that every partial sequence \( e \) is weakly \( \gamma_e \)-computable, tracked by the partial function \( n \mapsto \gamma v_i^{-1} \) (where \( \gamma v_i^{-1} \) is a \( \gamma \)-code for \( v_i \)), and \( e \) is \( \gamma_e \)-computable if, and only if, \( \text{dom}(e) \) is r.e. Normally, but not always, our sequences \( e \) will be total and hence \( \gamma_e \)-computable.

Theorem 3.2 [7] Let \( (A, \alpha) \) be a weakly effective partial \( \Sigma \)-algebra and let \( e: \mathbb{N} \to A \) be a partial sequence that is weakly \( \alpha \)-computable.

(i) Then \( \text{TE}_e: T(\Sigma, V) \to A \) is weakly \( (\gamma, \alpha) \)-computable and the inclusion \( \nu: (e) \to A \) is \( (\gamma_e, \alpha) \)-computable.

(ii) If \( (A, \alpha) \) is effective and \( e \) is \( \alpha \)-computable then \( \Omega_e \) is r.e. and \( (e), \gamma_e \) is effective.

Proposition 3.3 Let \( (A, \alpha) \) be an effective partial \( \Sigma \)-algebra where \( = \) is \( \alpha \)-semicomputable. If \( e \) is an \( \alpha \)-computable sequence such that \( (e) = A \) then \( \gamma_e \sim \alpha \).

Proof. Let \( \hat{\nu} \) be a partial recursive tracking function for the inclusion \( \nu: (e) \to A \). Then \( \hat{\nu} \) witnesses that \( \gamma_e \leq \alpha \). For the converse let \( \equiv_\alpha \) be an r.e. relation witnessing equality with respect to \( \alpha \). Note that \( \Omega_e \) is r.e. by Theorem 3.2. Define a partial function \( f \) by
\[
f(n) \simeq \nu m \in \Omega_e [n \equiv_\alpha \hat{\nu}(m)].
\]

Then \( f \) is partial recursive and a witness to \( \alpha \leq \gamma_e \).
The following is a version of Mal’cev’s result [2] for finitely generated partial $\Sigma$-algebras.

**Corollary 3.4** If $A$ is a finitely generated partial $\Sigma$-algebra then $A$ has, up to equivalence, at most one effective numbering with semicomputable equality.

**Proof.** Fix a generating set $\{x_1, \ldots, x_n\}$ of $A$ and define $e$ by $e(i) = x_i$ for $i = 1, \ldots, n$ and $e(i) = x_1$ for $i > n$. Then $\langle e \rangle = A$ and $e$ is $\alpha$-computable for any numbering $\alpha$ of $A$. Therefore, if $(A, \alpha)$ is effective with $\alpha$-semicomputable $= \alpha \sim \gamma_e$. \hfill \qed

### 4 Approximation-limit pair

Common algebras, such as the field of real numbers, are not finitely generated. Often they are generated by a limit process from a finitely generated subalgebra of approximations. This is the case for the reals, where any real number is the limit of some sequence of rational approximations. The approximating relation and the limit process, are obviously co-dependent. Therefore, we abstract these notions together in an approximation-limit pair.

**Definition 4.1** Let $A$ be a partial $\Sigma$-algebra and $e : \mathbb{N} \rightarrow A$ be a partial sequence.

(i) An approximation relation is a left-total relation $\text{aprx} \subseteq (A \times \mathbb{N}) \times \langle e \rangle$ (i.e., for each $x \in A$ and $n \in \mathbb{N}$ there exists $a \in \langle e \rangle$ satisfying the relation) such that for all $a \in \langle e \rangle$ and all $n \in \mathbb{N}$, $(a, n, a) \in \text{aprx}$.

(ii) A limit operation is a partial function $\lim : \langle e \rangle^{\mathbb{N}} \rightarrow A$ such that for all $a \in \langle e \rangle$, $\lim((a)) = a$.

(iii) An approximation sequence for $x \in A$ is a sequence $(a_n) \in \langle e \rangle^{\mathbb{N}}$ satisfying $(x, n, a_n) \in \text{aprx}$ for all $n \in \mathbb{N}$.

(iv) The pair $(\text{aprx}, \lim)$ is an approximation-limit pair for $A$ and $e$ if for each approximation sequence $(a_n)$ for $x$,

$$\lim((a_n)) \sim x.$$

If $(x, n, a_n) \in \text{aprx}$ it is helpful to think of $a_n$ as an $n^{th}$-level approximation of $x$. Although not strictly necessary, condition (i) includes the natural requirement that any approximation is an $n^{th}$-level approximation of itself, for all $n$. Furthermore, condition (ii) requires that $\lim$ behaves well with respect to constant sequences. This has the benefit that our computable completion of $\langle e \rangle$ will contain $\langle e \rangle$.

Note that the above definition does not relate the approximation-limit pair to the algebraic structure of $A$. The algebraic structure of $A$ is only used to generate the set $\langle e \rangle$ of approximations. Also note that the $\text{aprx}$ relation of an approximation-limit pair induces a natural topology, where the subbasic open sets are the sets of the form $B_{n,a} = \{x \in A : (x, n, a) \in \text{aprx}\}$ for all $n \in \mathbb{N}$ and $a \in \langle e \rangle$. In fact, we have chosen the terminology to reflect this fact. By the requirement that an element $a \in \langle e \rangle$ satisfies $(a, n, a) \in \text{aprx}$ for all $n$ it follows that $\langle e \rangle$ is dense in $A$.

For a numbered set $(A, \alpha)$ we denote the set of all $\alpha$-computable sequences in $A$ by $A^{\mathbb{N}}_{\alpha}$. A natural numbering $\alpha^{\ast}$ of $A^{\mathbb{N}}_{k,\alpha}$ obtained from $\alpha$ is defined by

$$\alpha^{\ast}(n) = \lambda k, \alpha(\varphi_n(k)),$$

for all $n$ such that $\alpha \circ \varphi_n$ is total.

Consider a partial $\Sigma$-algebra $A$ and a partial sequence $e : \mathbb{N} \rightarrow A$. Recall that $\langle (\langle e \rangle, \gamma_e) \rangle$ is a weakly effective partial $\Sigma$-algebra (effective if the operations are total) and $e$ is weakly $\gamma_e$-computable. We consider the numbered set $\langle (\langle e \rangle^{\mathbb{N}}_{k,\gamma_e}, \gamma_e^{\ast}) \rangle$ of sequences over approximations.

Let $A$ be a (possibly uncountable) partial $\Sigma$-algebra, $e : \mathbb{N} \rightarrow A$ be a partial sequence and $(\text{aprx}, \lim)$ be an approximation-limit pair for $A$ and $e$. We define a new standard numbering $\gamma_e$ of a subset $A_{k,\gamma_e}$ of $A$ depending on $(\text{aprx}, \lim)$ and $e$. Let

$$\Omega_{\gamma_e} = \{ n \in \Omega_{\gamma_e} : \lambda k, \gamma_e \varphi_n(k) \text{ is an approximation sequence}\},$$

and define $\gamma_e : \Omega_{\gamma_e} \rightarrow A$ by

$$\gamma_e(n) = \lim(\lambda k, \gamma_e \varphi_n(k)).$$
Definition 4.2 Let $A_{k, \gamma_e} = \bar{\gamma}_e[\Omega_{\gamma_e}]$. Elements of $A_{k, \gamma_e}$ are called computable, and the set $A_{k, \gamma_e}$ is the computable completion of $\langle e \rangle$ in $A$ with respect to $(\text{aprx}, \lim)$.

Proposition 4.3 $\langle e \rangle \subseteq A_{k, \gamma_e}$, and the inclusion $\iota : \langle e \rangle \to A_{k, \gamma_e}$ is $(\gamma_e, \bar{\gamma}_e)$-computable.

Proof. For $a \in \langle e \rangle$ the constant sequence $(a)$ is an approximation sequence for $a$ and $\lim((a)) = a$. There exists a primitive recursive function $s$ such that if $k$ is a $\gamma_e$-index of $a$ then $\varphi_s(k) = \lambda n.k$. Thus, $s(k)$ is a $\bar{\gamma}_e$-index of $a$.

Definition 4.4 Let $A$ be a partial $\Sigma$-algebra, $e : \mathbb{N} \to A$ a partial sequence and $(\text{aprx}, \lim)$ be an approximation-limit pair for $A$ and $e$. Let $\alpha$ and $\beta$ be numberings of $A_{k, \gamma_e}$. The approximation-limit pair $(\text{aprx}, \lim)$ for $A$ and $e$ is computable with respect to $\alpha$ and $\beta$ if

(i) $\text{aprx}$ has $(\alpha \times \text{id}, \gamma_e)$-computable selection, and

(ii) $\lim : \langle e \rangle_{k, \gamma_e}^N \to A_{k, \gamma_e}$ is weakly $(\gamma_e^*, \beta)$-computable.

When $\alpha = \beta$ in the definition we say that $(\text{aprx}, \lim)$ is computable with respect to $\alpha$.

Since the operation $\lim$ of a computable approximation-limit pair $(\text{aprx}, \lim)$ only need to be weakly effective we may without loss of generality assume that $\lim$ only is defined on approximation sequences.

Proposition 4.5 The approximation-limit pair $(\text{aprx}, \lim)$ is computable with respect to the standard numbering $\bar{\gamma}_e$ of $A_{k, \gamma_e}$.

Proof. If $n \in \Omega_{\gamma_e}$ then the sequence $\lambda k. \gamma_e \varphi_e(k)$ is an approximation sequence so the selection for $\text{aprx}$ is tracked by $\lambda n.k \gamma_e \varphi_n(k)$. The operation $\lim$ is weakly effective, tracked by the identity from $\Omega_{\gamma_e}$ to $\Omega_{\bar{\gamma}_e}$. In fact, as remarked above, these sets can be assumed to be equal.

Theorem 4.6 Let $A$ be a partial $\Sigma$-algebra, $e$ be a partial sequence in $A$ and $(\text{aprx}, \lim)$ be an approximation-limit pair for $A$ and $e$. Let $\alpha$ and $\beta$ be numberings of the set $A_{k, \gamma_e}$.

(i) If $\text{aprx}$ has $(\alpha \times \text{id}, \gamma_e)$-computable selection then $\alpha \leq \bar{\gamma}_e$.

(ii) If $\lim$ is weakly $(\gamma_e^*, \beta)$-computable then $\gamma_e \leq \beta$.

(iii) If $(\text{aprx}, \lim)$ is computable with respect to $\alpha$ and $\beta$ then $\alpha \leq \bar{\gamma}_e \leq \beta$.

Proof.

(i) Let $\text{aprx}$ track the selection for $\text{aprx}$. For $x \in A_{k, \gamma_e}$ let $n$ be such that $\alpha(n) \simeq x$. Then

$$\lambda k. \gamma_e \varphi_e(n, k) \in \langle e \rangle_{k, \gamma_e}^N.$$ 

Let $s$ be a primitive recursive function such that $\varphi_e(n, k) \simeq \varphi_s(n)(k)$ for each $n$ and $k$, i.e., $\gamma_e^*(s(n)) = \lambda k. \gamma_e \varphi_e(n, k)$. But that sequence is an approximation sequence so

$$\bar{\gamma}_e(s(n)) \simeq \lim(\lambda k. \gamma_e \varphi_e(n, k)) \simeq x$$

since we are dealing with an approximation-limit pair.

(ii) Let $n \in \Omega_{\gamma_e}$. Then $\bar{\gamma}_e(n) \downarrow$ and

$$\bar{\gamma}_e(n) \simeq \lim(\lambda k. \gamma_e \varphi_e(k))$$

$$\simeq \lim(\gamma_e^*(n))$$

$$\simeq \beta(\lim(n)),$$

where $\lim$ tracks $\lim$. Thus $\bar{\gamma}_e \leq \beta$, witnessed by $\lim$.

(iii) By (i) and (ii).
Corollary 4.7 If \((\text{aprx}, \lim)\) is computable with respect to a numbering \(\alpha\) of \(A_{k, \gamma_e}\) then \(\gamma_e \sim \alpha\). Thus, up to recursive equivalence, there is exactly one numbering of the set \(A_{k, \gamma_e}\) for which the approximation-limit pair \((\text{aprx}, \lim)\) is computable.

It is often the case in applications that every effective numbering \(\alpha\) of a partial \(\Sigma\)-algebra \(A\) makes it possible to define \(\text{aprx}\) with \((\alpha \times \text{id}, \gamma_e)\)-computable selection. Put differently, \(\Sigma\) often includes operations sufficient for computably tracking selection for \(\text{aprx}\) for all effective numberings of \(A\).

We now consider sufficient conditions for \(\gamma_e\) being an effective numbering of \(A_{k, \gamma_e}\).

Proposition 4.8 If \(A_{k, \gamma_e}\) is a partial \(\Sigma\)-algebra with an effective numbering \(\alpha\) making \((\text{aprx}, \lim)\) computable then \(\gamma_e \sim \alpha\) and hence \(\gamma_e\) is effective.

Proof. By Corollary 4.7.

Without a priori access to such a numbering we need a notion of effective continuity with respect to an approximation relation.

Again we consider a partial \(\Sigma\)-algebra \(A\), a partial sequence \(e: \mathbb{N} \to A\) and an approximation relation \(\text{aprx}\) with respect to \(A\) and \(e\). We temporarily adopt the notations \(m = m_1, \ldots, m_n\) and \(\gamma_e(m) = \gamma_e(m_1), \ldots, \gamma_e(m_n)\).

Definition 4.9 Let \(\sigma\) be a partial \(n\)-ary operation of \(A\). Then \(\sigma\) is effectively continuous with respect to \(\text{aprx}\) if there is an \((n + 1)\)-ary partial recursive function \(t\) such that whenever \(\sigma(\gamma_e(m))\) \(\downarrow\) then

(i) \(t(m, k)\) \(\downarrow\) for each \(k\), and
(ii) if \((\gamma_e(m_i), t(m, k), a_{i,k}) \in \text{aprx}\) for \(i = 1, \ldots, n\) then

\[ \sigma(\gamma_e(m)), k, \sigma(a_{1,k}, \ldots, a_{n,k}) \in \text{aprx}. \]

Let the partial operation \(\sigma\) be effectively continuous with witness \(t\) and let \(\hat{\sigma}\) track \(\sigma\) on \(\langle e \rangle\) with respect to \(\gamma_e\). Assume \(m \in \Omega_{\gamma_e}\) and that \(\sigma(\gamma_e(m))\) \(\downarrow\). Then for each \(i\), \(\gamma_e(m_i) = \lambda k: \gamma_e(\varphi_{m_i}, (k)\) is an approximation sequence, i.e.,

\[ (\gamma_e(m_i), k, \gamma_e(\varphi_{m_i}(k)) \in \text{aprx}, \]

for each \(k \in \mathbb{N}\).

Fix \(k \in \mathbb{N}\). Note that \(t(m, k)\) is defined. Thus, we have

\[ (\gamma_e(m_i), t(m, k), \gamma_e(\varphi_{m_i}(t(m, k)) \in \text{aprx}, \]

for each \(i\). By \(\sigma\) being effectively continuous we obtain

\[ \sigma(\gamma_e(m)), k, \sigma(\gamma_e(\varphi_{m_1}(t(m, k))), \ldots, \gamma_e(\varphi_{m_n}(t(m, k))) \in \text{aprx}. \]

Since \(\hat{\sigma}\) tracks \(\sigma\) we have

\[ \sigma(\gamma_e(\varphi_{m_1}(t(m, k))), \ldots, \gamma_e(\varphi_{m_n}(t(m, k)))) \]

\[ \sim \gamma_e(\varphi_{m_1}(t(m, k))), \ldots, \varphi_{m_n}(t(m, k))). \]

It follows that

\[ \lambda k: \gamma_e(\varphi_{m_1}(t(m, k))), \ldots, \varphi_{m_n}(t(m, k))) \]

is a \(\gamma_e\)-computable approximation sequence. Let \(s\) be a primitive recursive function obtained from the s-m-n theorem such that

\[ \hat{\sigma}(\varphi_{m_1}(t(m, k))), \ldots, \varphi_{m_n}(t(m, k))) \sim \varphi_{s(m)}(k) \]
for all \( \mathbf{m} \) and \( k \). Then \( \lambda k.\gamma_e \varphi_s(\mathbf{m})(k) \) is a \( \gamma_e \)-computable approximation sequence obtained from \( \sigma(\bar{\gamma}_e(\mathbf{m})) \). Now, if \((\text{aprx}, \lim)\) is an approximation-limit pair and \( \sigma \) is effectively continuous with respect to \( \text{aprx} \) then

\[
\sigma(\bar{\gamma}_e(\mathbf{m})) \simeq \lim (\lambda k.\gamma_e \varphi_s(\mathbf{m})(k)) \\
\simeq \gamma_e(s(\mathbf{m})).
\]

Thus, \( \sigma \) is tracked by \( s \) with respect to \( \gamma_e \) showing that \( A_{k,\gamma_e} \) is closed under \( \sigma \), and \( \sigma \) is weakly \((\bar{\gamma}_e^n, \bar{\gamma}_e)\)-computable.

We have proved the following result:

**Theorem 4.10** Let \( A \) be a partial \( \Sigma \)-algebra, \( e : N \to A \) be a partial sequence and \((\text{aprx}, \lim)\) be an approximation-limit pair for \( A \) and \( e \). Assume that every operation of \( \Sigma \) is effectively continuous on \( A_{k,\gamma_e} \) with respect to \((\text{aprx}, \lim)\).

(i) Then \((A_{k,\gamma_e}, \bar{\gamma}_e)\) is a weakly effective partial \( \Sigma \)-algebra.
(ii) The inclusion \( \iota : \langle e \rangle \to A_{k,\gamma_e} \) is a \((\gamma_e, \bar{\gamma}_e)\)-computable \( \Sigma \)-embedding.

In case each operation of \( A \) is total then the conclusion of the theorem states that \((A_{k,\gamma_e}, \bar{\gamma}_e)\) is effective.

The use of \( \gamma_e \) is not restrictive. Each weakly effective numbering \( \alpha \) is equivalent to \( \gamma_e \) for some \( e \).

**Proposition 4.11** Let \((A, \alpha)\) be a weakly effective partial \( \Sigma \)-algebra. Then \( \alpha \sim \gamma_e \) for some partial sequence \( e \).

**Proof.** We let \( e = \alpha \). Thus, \( \langle e \rangle = A \), \( e \) is weakly \( \alpha \)-computable, and hence the inclusion \( \iota : \langle e \rangle \to A \) is \((\gamma_e, \alpha)\)-computable, i.e., \( \gamma_e \leq \alpha \). For the converse reduction we have for \( n \in \Omega_\gamma \),

\[
\alpha(n) \simeq e(n) \simeq \text{TE}_\gamma(v_n) \simeq \gamma_e(\bar{\gamma}_e(v_n)),
\]

i.e., the recursive function \( n \mapsto \bar{\gamma}_e(v_n) \) witnesses that \( \alpha \leq \gamma_e \).

\[ \square \]

## 5 Applications

### 5.1 Real numbers

We revisit the ordered field of real numbers. All results herein are known, but we point out how our general approach relates to the traditional developments of results about real numbers.

Let \( \mathbb{R} \) be the partial algebra \( \mathbb{R} = (\mathbb{R}; 0, 1, +, -, \times, (\cdot)^{-1}, \chi_\prec) \), where \( \chi_\prec \) is the partial strict ordering relation. Let \( \Sigma \) be its signature, and let \( e : N \to \mathbb{R} \) be the constant function with value 0. Thus, \( \langle e \rangle = \mathbb{Q} \). The rationals form an effective \( \Sigma \)-subalgebra under the standard numbering \( \gamma_e \), with \( = \) and \( < \gamma_e \)-decidable. By Proposition 3.3, \( \gamma_e \) is up to recursive equivalence the only effective numbering of \( \mathbb{Q} \) with semicomputable equality.

Let \( \text{aprx} \subseteq \mathbb{R} \times \mathbb{N} \times \langle e \rangle \) be defined by

\[
(x, k, a) \in \text{aprx} \iff |x - a| < 2^{-k}.
\]

An approximation sequence for \( \text{aprx} \) is what is known in the literature as a fast Cauchy sequence. Let the limit operation \( \lim \) take a fast Cauchy sequence to its Cauchy limit. The pair \((\text{aprx}, \lim)\) is then an approximation-limit pair for \( \mathbb{R} \) and \( \langle e \rangle \).

The computable completion \( \mathbb{R}_{\gamma_e} \) of \( \langle e \rangle \) with respect to \((\text{aprx}, \lim)\) (in the sequel denoted by \( \mathbb{R}_k \)) is the set of computable real numbers. By Corollary 4.7 any numbering \( \alpha \) of \( \mathbb{R}_k \) making \((\text{aprx}, \lim)\) computable is recursively equivalent to the standard numbering \( \bar{\gamma}_e \).

**Lemma 5.1** If the partial \( \Sigma \)-algebra \((\mathbb{R}_k, \alpha)\) is effective then \( \text{aprx} \) has \((\alpha \times \text{id}, \gamma_e)\)-computable selection.

**Proof.** We use Proposition 2.7. First,

\[
(x, n, a) \in \text{aprx} \iff |x - a| < 2^{-n} \\
\iff x - a < 2^{-n} \land a - x < 2^{-n},
\]

is \((\bar{\gamma}_e \times \text{id}, \gamma_e)\)-semicomputable, and also \( \Omega_e \) is r.e., by Theorem 3.2 (ii), so the result follows. \[ \square \]
Corollary 5.2 If the partial $\Sigma$-algebra $(\mathbb{R}_k, \alpha)$ is effective then $\alpha \leq \bar{\gamma}_e$.

Proof. By Lemma 5.1 and Theorem 4.6 (i).

In our treatment so far we do not know if $\bar{\gamma}_e$ is an effective numbering of the $\Sigma$-algebra $\mathbb{R}_k$, in fact, we have not established that $\mathbb{R}_k$ is a $\Sigma$-algebra, i.e., closed under the operations in $\Sigma$. At this point there are two routes presenting themselves. Either show that all operations of $\Sigma$ are effectively continuous with respect to $\gamma_e$ and using Theorem 4.10, or show that there exists an effective numbering $\alpha$ of the $\Sigma$-algebra $\mathbb{R}_k$ such that (aprx, lim) is computable with respect to $\alpha$ and using Proposition 4.8.

Since it is routine to verify the effective continuity of the operations we will choose the former route. We only present here the argument that the domain of the partial operations, $\chi<\gamma$ and $(\cdot)^{-1}$, are $\gamma_e$-semicomputable. We first note that $<\gamma$ is $\gamma_e$-semicomputable. For if $(a_k)$ and $(b_k)$ are approximation sequences in $\langle e \rangle$ then

$$\lim((a_k)) < \lim((b_k)) \iff \exists k (b_k - a_k > 2^{-k+1}).$$

Let $W(m, n) \iff \exists k (\gamma_e \varphi_n (k) - \gamma_e \varphi_m (k) > 2^{-k+1})$. Then $W$ is r.e., and witnesses that $<\gamma$ is $\gamma_e$-semicomputable and hence that $\chi<\gamma$ is a $\gamma_e$-computable partial function. It also follows that $x \neq 0$ is $\gamma_e$-semicomputable so that $(\cdot)^{-1}$ is $\gamma_e$-computable by [9, Lemma 2.12].

Proposition 5.3 The ordered field $(\mathbb{R}_k, \gamma_e)$ is an effective partial $\Sigma$-algebra.

Let LIM be the function taking an $\alpha$-computable fast Cauchy sequence in $\mathbb{R}_k$ to its limit. We say that $(\mathbb{R}_k, \alpha)$ has a limit algorithm if LIM is weakly $(\alpha^{*}, \alpha)$-computable. A limit algorithm LIM is the traditional concept for computable structures, cf. Moschovakis [11], namely an algorithm computing limits for a class of converging computable sequences over the structure. On the other hand, the limit operation of an approximation-limit pair only applies to sequences over the finitely generated subalgebra of approximations.

Proposition 5.4 $(\mathbb{R}_k, \bar{\gamma}_e)$ has a limit algorithm.

Proof. A $\bar{\gamma}_e$-computable fast Cauchy sequence $(x_n)$ in $\mathbb{R}_k$ is a $\gamma_e$-computable double sequence $(a_{n,m})$ in $\langle e \rangle$. It is routine to verify that the diagonal sequence $(a_{n+1, n+1})$ is a $\gamma_e$-computable fast Cauchy sequence with an index obtained uniformly from a $\bar{\gamma}_e$-index of the original sequence and that $\mathrm{LIM}((x_n)) \simeq \lim((a_{n+1, n+1}))$.

This is our version of the Moschovakis–Hertling result on the stability of the representation of the reals.

Theorem 5.5 Let $\mathbb{R}_k = (\mathbb{R}_k; 0, 1, +, -, \times, (\cdot)^{-1}, \chi<\gamma)$ be the computable completion with respect to (aprx, lim) and $e$ and let $\alpha$ be an effective numbering of $\mathbb{R}_k$. Then the following hold.

(i) $\alpha \leq \bar{\gamma}_e$.

(ii) If (aprx, lim) is computable with respect to $\alpha$ then $\alpha \sim \bar{\gamma}_e$.

(iii) $(\mathbb{R}_k, \alpha)$ has a limit algorithm if, and only if, $\alpha \sim \bar{\gamma}_e$.

Proof.

(i) By Corollary 5.2.

(ii) By Theorem 4.6(ii).

(iii) If $\alpha \sim \bar{\gamma}_e$ then $(\mathbb{R}_k, \alpha)$ has a limit algorithm since $(\mathbb{R}_k, \bar{\gamma}_e)$ does. For the converse reduction assume $(\mathbb{R}_k, \alpha)$ has a limit algorithm. By (ii) it suffices to show that $\ell$ is weakly $(\gamma_e^{*}, \alpha)$-computable. The embedding $i: \langle e \rangle \to \mathbb{R}_k$ is $(\gamma_e, \alpha)$-computable by Theorem 3.2; let $i$ be the partial recursive tracking function. Let $n \in \Omega_{\bar{\gamma}_e}$, i.e., $\lambda k. \gamma_e \varphi_n (k)$ is a Cauchy sequence with respect to (aprx, lim). Let $s$ be a primitive recursive function such that $\varphi_{n}(k) \simeq i\varphi_n (k)$ for each $n$ and $k$ and let $\hat{\mathrm{LIM}}$ be a partial recursive tracking function for LIM. Then

$$\lim (\lambda k. \gamma_e \varphi_n (k)) \simeq \mathrm{LIM} (\lambda k. \alpha \varphi_{n}(k)) \simeq \alpha \hat{\mathrm{LIM}}(s(n))$$

and hence $\ell$ is weakly $(\gamma_e^{*}, \alpha)$-computable.
We see that in the class of effective numberings of $\mathbb{R}_k$ there is a largest numbering (up to recursive equivalence) and it is characterised by having a limit algorithm. Such a numbering is said to be a standard numbering of the recursive reals. Thus, a standard numbering is determined from the indexing of fast Cauchy sequences in $\mathbb{Q}$.

The ordering of the reals is not used in the proof of Theorem 5.5(i), in fact, if Corollary 4.7 is applied directly to the unordered reals, then the numbering is still unique. This is not really a strengthening since the ordering can be recovered from our chosen approximation relation $\text{aprx}$.

The algebraic formulation of Theorem 4.6 makes it easy to translate results to slightly different algebras. An easy exercise is to modify Theorem 5.5 to $\mathbb{R}' = (\mathbb{R}; 0, \frac{1}{2}, 1, +, -, \times, \chi_\infty)$ of real numbers. Note that $\Sigma_D$ does not contain the partial operation $(\cdot)^{-1}$ but has a new constant, $\frac{1}{2}$. The dyadic numbers are generated directly by the signature $\Sigma_D$, so we can let $e$ be the sequence that is constantly 0. Then $\gamma_e$ is a computable numbering of the $\Sigma_D$-algebra $\mathbb{D}$.

**Theorem 5.6** Let $\mathbb{R}_k' = (\mathbb{R}_k; 0, \frac{1}{2}, 1, +, -, \times, \chi_\infty)$ be the computable completion with respect to $(\text{aprx}, \lim)$ and $e$ and let $\alpha$ be an effective numbering of $\mathbb{R}_k'$. Then the following hold.

(i) $\alpha \leq \gamma_e$.

(ii) If $(\text{aprx}, \lim)$ is computable with respect to $\alpha$ then $\alpha \sim \gamma_e$.

(iii) $(\mathbb{R}_k', \alpha)$ has a limit algorithm if, and only if, $\alpha \sim \gamma_e$.

### 5.2 Inverse limits

Taking the inverse limit of an inverse system of algebras is an important “completion” process in mathematics and in modelling computations. A natural approximation-limit pair for this construction is obtained using the projection functions, providing a tool for analysing its effective content.

We consider inverse limits obtained from an algebra via a family of separating congruences. The more general situation of the inverse limit of an inverse system can be handled similarly only with a little more notational simplicity. In this section we assume the operations of $A$ to be total. Then $\{\equiv_n\}_n$ is a family of separating congruences on $A$ if each $\equiv_n$ is a congruence on $A$ and the following hold:

(i) $x \equiv_{n+1} y \implies x \equiv_n y$, and

(ii) $\bigcap_{n \in \mathbb{N}} \equiv_n = \{(x, x) : x \in A\}$.

We set $A_n = A/\equiv_n$ and let $\nu_n : A \to A_n$ be the factoring epimorphism, i.e., $\nu_n(a) = [a]_n = \{b \in A : a \equiv_n b\}$. For $n \geq m$ we let $\phi_m^n : A_n \to A_m$ be the epimorphism given by $[a]_n \mapsto [a]_m$. Then $(A_n, \phi_m^n)_{n \geq m}$ is an inverse system. Its inverse limit is denoted by $\varprojlim (A_n, \phi_m^n)$, or simply $\varprojlim A_n$ or $\hat{A}$, along with the epimorphisms $\tilde{\nu}_n : \hat{A} \to A_n$ satisfying $\tilde{\nu}_n \circ \phi_m^n = \tilde{\nu}_m$ for each $n \geq m$.

Recall that $\hat{A}$ is, up to isomorphism, the $\Sigma$-algebra

$$\hat{A} = \left\{ (a_n) \in \prod_{n = 0}^{\infty} A_n : \forall n (\phi_n^{n+1} (a_{n+1}) = a_n) \right\}$$

along with $\tilde{\nu}_n : \hat{A} \to A_n$ given by $\tilde{\nu}_n([a_k]) = a_n$, and the operations on $\hat{A}$ act pointwise. By the universal property of inverse limits there is a unique embedding $\theta : A \to \hat{A}$ such that $\phi_n \circ \theta = \nu_n$ for each $n$. It is given by $\theta(a) = ([a]_n)$.

Throughout we now let $(A, \alpha)$ be an effective $\Sigma$-algebra with a family $\{\equiv_n\}_n$ of separating congruences on $A$. Note that no assumptions are made on the effectivity of $\equiv_n$. We define a numbering $\alpha_n : \Omega_n \to A_n$ by $\alpha_n(k) = [\alpha(k)]_n$. Then $(A_n, \alpha_n)$ is an effective $\Sigma$-algebra since $\equiv_n$ is a congruence, and $\nu_n$ is $(\alpha, \alpha_n)$-computable, tracked by the identity.

By Proposition 4.11 there is a weakly $\alpha$-computable partial sequence $e$ in $A$ such that $\alpha$ is recursively equivalent to $\gamma_e$. Thus, we may use the results from Section 4 with $\alpha$ playing the role of $\gamma_e$ and $\tilde{\alpha}$ of $\gamma_e$. Furthermore, for notational simplicity, we will consider an approximation relation taking values in $A$ rather than in its isomorphic image $\theta(A)$.
Define a relation \( \text{aprx} \subseteq \tilde{A} \times \mathbb{N} \times A \) by
\[
(x, n, a) \in \text{aprx} \iff \tilde{\phi}_n (x) = v_n (a).
\]
Then define the partial function \( \lim : A^\mathbb{N} \to \tilde{A} \) as follows. Suppose \( (a_n) \) is an approximation sequence in \( A \) with respect to \( \text{aprx} \) and induced by \( x \). Then
\[
\lim ((a_n)) \simeq ([a_n]) = x.
\]
It follows that \((\text{aprx}, \lim)\) is an approximation-limit pair for \( A \) and \( \alpha \).

Let \( \tilde{A}_{k, \alpha} \) be the computable completion of \( A \) and \( \alpha \). Then \( (\tilde{A}_{k, \alpha}, \tilde{\alpha}) \) is a numbered set (where \( \tilde{\alpha} \) plays the role of \( \tilde{\gamma} \), i.e., \( \tilde{\alpha}(n) \simeq \lambda k. v_k \alpha \varphi_n (k) \) when \( \lambda k. \alpha \varphi_n (k) \) is an approximation sequence). Let \( \tilde{\phi}_n \) be the restriction of \( \tilde{\phi}_n \) to \( \tilde{A}_{k, \alpha} \).

**Proposition 5.7**

(i) \( \theta : A \to \tilde{A}_{k, \alpha} \) is \((\alpha, \tilde{\alpha})\)-computable.

(ii) \( \phi_n : \tilde{A}_{k, \alpha} \to A_n \) is \((\tilde{\alpha}, \alpha_n)\)-computable, uniformly in \( n \).

(iii) \( (\tilde{A}_{k, \alpha}, \tilde{\alpha}) \) is an effective \( \Sigma \)-algebra.

**Proof.**

(i) Let \( s \) be primitive recursive such that \( \varphi_{s(n)} \) is the constant function with value \( n \). For \( n \in \Omega_\alpha \), we then have
\[
\theta(\alpha(n)) = \lambda k. v_k \alpha \varphi_{s(n)}(k) = \tilde{\alpha}(s(n)).
\]

(ii) Let \( t \) be a primitive recursive function such that \( \varphi_{t(m)}(n) \simeq \varphi_{t(n)}(m) \) for each \( m \) and \( n \). Then for \( m \in \Omega_\alpha \),
\[
\phi_n (\tilde{\alpha}(m)) = \phi_n (\lambda k. v_k \alpha \varphi_{s(n)}(k)) = \phi_n (\lambda k. v_k \alpha \varphi_{t(n)}(m)) = \alpha_n (\varphi_{t(n)}(m)),
\]
i.e., \( \varphi_{t(n)} \) tracks \( \phi_n \).

(iii) It is well-known that \( \tilde{A} \) is a \( \Sigma \)-algebra. Let \( \sigma \in \Sigma \) be \( k \)-ary and let \( \sigma_{\tilde{A}_{k, \alpha}} \) be the restriction of \( \sigma \) to \( \tilde{A}_{k, \alpha} \).

Now, \( \equiv_n \) is a congruence relation and hence \( \sigma_{\tilde{A}_{k, \alpha}} \) is effectively continuous with respect to \((\text{aprx}, \lim)\) by letting \( t(m_1, \ldots, m_k, n) = n \). Thus, \((\tilde{A}_{k, \alpha}, \tilde{\alpha})\) is an effective \( \Sigma \)-algebra by Theorem 4.10. \( \square \)

A sequence \((a_n)\) in \( A \) is said to be a Cauchy sequence if \( a_{n+1} \equiv_n a_n \) for each \( n \). Note that this corresponds to being an approximation sequence with respect to \( \text{aprx} \). Similarly, a sequence \((x_n)\) in \( \tilde{A} \) is said to be a Cauchy sequence if \( \tilde{\phi}_n (x_{n+1}) = \tilde{\phi}_n (x_n) \) for each \( n \). Thus, if \( x_n = ([a_{nk}]_k) \), where \( a_{nk} \in A \), then it is required that \( a_{n+1,n} \equiv_n a_{nn} \) for each \( n \). Using the above notation let the partial function \( \text{LIM} : A^\mathbb{N} \to \tilde{A} \) be defined on each Cauchy sequence \((x_n)\) by
\[
\text{LIM}((x_n)) \simeq \lim((a_{nn})).
\]
Note that \((a_{nn})\) is a Cauchy sequence in \( A \) since
\[
a_{n+1,n+1} \equiv_n a_{n+1,n} \equiv_n a_{nn},
\]
and therefore \( \text{LIM}((x_n)) \) is defined. \( \text{LIM} \) is the limit operator with respect to the natural ultrametric on \( \tilde{A} \) induced by \( \{\equiv_n\}_n \). This will be further discussed in Section 5.3.

Let \( \beta \) be a numbering of \( \tilde{A}_{k, \alpha} \). We say that \((\tilde{A}_{k, \alpha}, \beta)\) has a limit algorithm if \( \text{LIM} \) is weakly \((\beta^*, \beta^r)\)-computable.

**Proposition 5.8** \((\tilde{A}_{k, \alpha}, \tilde{\alpha})\) has a limit algorithm.

**Proof.** Assume \( \text{LIM}(\alpha^*(m)) \). Then \( \alpha^*(m) = \lambda k. \tilde{\alpha} \varphi_{n}(k) \) is a Cauchy sequence in \( \tilde{A}_{k, \alpha} \) and for each \( k \), \( \tilde{\alpha} \varphi_{n}(k) = \lambda \lambda \alpha \varphi_{n} (k) (l) \) is a Cauchy sequence in \( A \). Thus, the diagonal sequence is \( \lambda k. \alpha \varphi_{n} (k) (k) \). Let \( s \) be a primitive recursive function such that \( \varphi_{n} (k) (k) \simeq \varphi_{s(n)} (k) \) for each \( k \). Then
\[
\text{LIM}(\alpha^*(m)) \simeq \lim(\lambda k. \alpha \varphi_{s(n)} (k)) \simeq \tilde{\alpha}(s(m)). \] \( \square \)
Theorem 5.9 Let \((A, \alpha)\) be an effective \(\Sigma\)-algebra and let \(\{\equiv_n\}_n\) be a family of separating congruences on \(A\). Let \(\beta\) be a numbering of \(\tilde{A}_{k, \alpha}\) such that \(\phi_n : \tilde{A}_{k, \alpha} \to A_n\) is \((\beta, \alpha_n)\)-computable, uniformly in \(n\). Then

(i) \(\beta \leq \bar{\alpha}\).
(ii) If the embedding \(\theta : A \to \tilde{A}_{k, \alpha}\) is \((\alpha, \beta)\)-computable then \(\bar{\beta} \sim \bar{\alpha}\) if, and only if, \((\tilde{A}_{k, \alpha}, \beta)\) has a limit algorithm.

Proof.

(i) Let \(t\) be a total recursive function such that \(\varphi_{t(n)}\) tracks \(\phi_n\) with respect to \(\beta\) and \(\alpha_n\). Then

\[
\text{apr}(m, n) \simeq \varphi_{t(n)}(m),
\]

witnesses that \(\text{apr}\) has \((\beta \times \text{id}, \alpha)\)-computable selection. It follows by Theorem 4.6 (i) that \(\beta \leq \bar{\alpha}\).

(ii) If \(\beta \sim \bar{\alpha}\) then \((\tilde{A}_{k, \alpha}, \beta)\) has a limit algorithm by Proposition 5.8. To show the converse it suffices by Theorem 4.6 (ii) to show that \(\text{lim} \beta = \bar{\alpha}\). Suppose \((\tilde{A}_{k, \alpha}, \beta)\) has a limit algorithm where the partial recursive function \(\text{LIM}\) tracks \(\text{LIM}\). Note that if \((a_n)\) is a Cauchy sequence in \(A\) then

\[
\text{lim}(\langle a_n \rangle) \simeq \text{LIM}(\langle \theta(a_n) \rangle).
\]

Assume \(m \in \Omega_{\alpha^*} = \Omega_{\alpha}, \) i.e., \(\alpha^*(m) = \lambda k. \alpha \varphi_m(k)\). Let \(\tilde{\theta}\) be a partial recursive function tracking \(\theta\) with respect to \(\alpha\) and \(\beta\) and let \(s\) be primitive recursive such that \(\varphi_{\tilde{\theta}}(m) \simeq \varphi_s(m)\) for each \(m\) and \(k\). Then

\[
\begin{align*}
\text{lim}(\lambda k. \alpha \varphi_m(k)) & \simeq \text{LIM}(\lambda k. \beta \varphi_m(k)) \\
& \simeq \text{LIM}(\lambda k. \beta \varphi_s(m)(k)) \\
& \simeq \beta(\text{LIM}(s(m))).
\end{align*}
\]

We conclude with some remarks on the complexity of \(\equiv_n\) on \(A\). The congruence relation \(\equiv_n\) on \(A\) is extended to \(\tilde{A}\) by, for \(x, y \in A\),

\[
x \equiv_n y \iff \tilde{\phi}_n(x) = \tilde{\phi}_n(y).
\]

Proposition 5.10 Let \((A, \alpha)\) be an effective \(\Sigma\)-algebra with a family \(\{\equiv_n\}_n\) of separating congruences, assume \(\Omega_{\alpha}\) is r.e., and let \(\beta\) be a numbering of \(\tilde{A}_{k, \alpha}\).

(i) Assume \(\theta : A \to \tilde{A}_{k, \alpha}\) is \((\alpha, \beta)\)-computable. If \(\{\equiv_n\}_n\) on \(\tilde{A}_{k, \alpha}\) is \(\beta\)-semicomputable, uniformly in \(n\), then \(\phi_n\) is \((\beta, \alpha_n)\)-computable, uniformly in \(n\).

(ii) If \(\phi_n\) is \((\beta, \alpha_n)\)-computable and \(\{\equiv_n\}_n\) on \(A\) is \(\alpha\)-semicomputable, uniformly in \(n\), then \(\{\equiv_n\}_n\) on \(\tilde{A}_{k, \alpha}\) is \(\beta\)-semicomputable, uniformly in \(n\).

Proof.

(i) The partial recursive function \(\text{apr}(m) = \nu k \in \Omega_{\alpha} \mid \theta\alpha(k) \equiv_n \beta(m)\) tracks \(\phi_n\).

(ii) Let \(\text{apr}\) be the partial recursive function from the proof of Theorem 5.9 tracking the computable selection for \(\text{apr}\) with respect to \(\beta\). Then for \(i, j \in \Omega_\beta\),

\[
\begin{align*}
\beta(i) \equiv_n \beta(j) \iff \phi_n(\beta(i)) = \phi_n(\beta(j)) \\
\iff \alpha(\text{apr}(i, n)) \equiv_n \alpha(\text{apr}(j, n)).
\end{align*}
\]

Thus, in the event that \(\{\equiv_n\}_n\) on \(A\) is \(\alpha\)-semicomputable, uniformly in \(n\), and \(\Omega_{\alpha}\) is r.e., then \(\phi_n\) is \((\beta, \alpha_n)\)-computable, uniformly in \(n\), if, and only if, \(\{\equiv_n\}_n\) is \(\beta\)-semicomputable, uniformly in \(n\).
5.3 Effective metric partial $\Sigma$-algebras

A metric space $M$ with metric $d$ is effective with respect to a numbering $\alpha$ of $M$ if $d$ is $(\alpha^2, \rho)$-computable, where $\rho$ is a standard numbering of the computable reals $\mathbb{R}_k$ as in Section 5.1.

A metric partial $\Sigma$-algebra is a partial $\Sigma$-algebra $A$ equipped with a metric $d : A^2 \rightarrow \mathbb{R}$ such that each partial operation of $A$ is continuous. We say that $A$ is (weakly) effective with respect to a numbering $\alpha$ if $(A, \alpha)$ is an effective metric space and each operation of $A$ is (weakly) effective.

Now let $(A, \alpha)$ be an effective metric partial $\Sigma$-algebra and suppose $e$ is an $\alpha$-computable sequence such that $\langle e \rangle$ is dense in $A$. Then, by Theorem 5.2, $\langle e \rangle$ is a dense partial $\Sigma$-subalgebra effective under the numbering $\gamma_e$, the inclusion $\iota : \langle e \rangle \rightarrow A$ is $(\gamma_e, \alpha)$-computable, and $\Omega_e$ is r.e.

Define a relation $\text{aprx} \subseteq A \times \mathbb{N} \times \langle e \rangle$ by

\[ (x, n, a) \in \text{aprx} \iff d(x, a) < 2^{-n}. \]

An approximation sequence with respect to $\text{aprx}$ is thus a fast Cauchy sequence with a limit in $A$. Let $\lim : \langle e \rangle^{\mathbb{N}} \rightarrow A$ be the corresponding limit function defined by $\lim((a_n)) \simeq x$ whenever $(a_n)$ is an approximation sequence with respect to $\text{aprx}$ and generated by $x \in A$. Thus, $\lim$ is well-defined.

The relation $\text{aprx}$ is $(\alpha \times \mathbb{N} \times \gamma_e)$-semicomputable since $< \rho$-semicomputable, using the computability of $d$ and $\alpha$, and it is left-total since $\langle e \rangle$ is dense in $A$. Hence $\text{aprx}$ has computable selection by Proposition 2.7.

It now follows from Theorem 4.6 that $\bar{\gamma}_e$ is a numbering of $A$ and that $\alpha \leq \bar{\gamma}_e$. Furthermore, if $\lim$ is weakly $(\bar{\gamma}_e, \alpha)$-computable then $\alpha \sim \bar{\gamma}_e$.

We say that $(A, \alpha)$ has a limit algorithm if the function $\text{LIM}$ taking fast $\alpha$-computable Cauchy sequences in $A$ to its limit in $A$, if it exists, is weakly $(\alpha^*, \alpha)$-computable. Using the $(\bar{\gamma}_e, \alpha)$-computable inclusion $\iota : \langle e \rangle \rightarrow A$ it follows that $\lim$ is $(\bar{\gamma}_e, \alpha)$-computable if $(A, \alpha)$ has a limit algorithm.

Using the above observations we obtain the following theorem with a proof completely analogous to that of Theorem 5.5.

**Theorem 5.11** Let $(A, \alpha)$ be an effective metric partial $\Sigma$-algebra and let $e$ be an $\alpha$-computable sequence such that $\langle e \rangle$ is dense in $A$.

(i) $\bar{\gamma}_e$ is a numbering of $A$ and $\alpha \leq \bar{\gamma}_e$.
(ii) If $\lim$ is weakly $(\bar{\gamma}_e^*, \alpha)$-computable then $\alpha \sim \bar{\gamma}_e$.
(iii) $(A, \alpha)$ has a limit algorithm if, and only if, $\alpha \sim \bar{\gamma}_e$.

An important role of $\alpha$ in the above was to guarantee that the metric $d$ was computable on $\langle e \rangle$ with respect to $\gamma_e$. Suppose $A$ is a metric partial $\Sigma$-algebra and $e$ is a sequence in $A$ such that the metric $d$ restricted to $\langle e \rangle$ is $(\gamma_e, \rho)$-computable. Then we may consider the recursive closure $\langle e \rangle^{A}_{\gamma_e}$ of $\langle e \rangle$ in $A$ with respect to $\text{aprx}$ defined above. It is straightforward to verify that $d$ restricted to $\langle e \rangle^{A}_{\gamma_e}$ is $(\gamma_e, \rho)$-computable, using the triangle inequality of the metric. If the operations are effectively continuous with respect to $(\text{aprx}, \lim)$ then we obtain the computable completion $\langle e \rangle^{A}_{\gamma_e}$ as a weakly effective metric partial $\Sigma$-algebra and its numbering $\bar{\gamma}_e$ is unique up to recursive equivalence having a limit algorithm. As an example, the ordered field $\mathbb{R}_k$ of recursive reals with its standard numbering was obtained in this way. For further results in this direction we refer to [9].

To conclude this section we return to the inverse limit construction of Section 5.2. Let $A$ be a $\Sigma$-algebra with a family of separating congruences $\left\{ \equiv_n \right\}_n$. There is a natural metric $d$ on $A$ defined by

\[ d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 2^{-n_0} & \text{where } n_0 \text{ is the least } n \text{ such that } x \equiv_n y. \end{cases} \]

In fact, $d$ is an ultrametric. Furthermore, each ultrametric $\Sigma$-algebra with non-expansive operations can be construed as a $\Sigma$-algebra with a family of separating congruences $\left\{ \equiv_n \right\}_n$ up to topological equivalence (i.e., homeomorphic algebras).

It is natural to ask when the computable completion for the inverse limit as in Section 5.2 corresponds to the computable completion as a metric algebra. That is, are there reasonable conditions for when the metric $d$ is computable?
Theorem 5.12 Let \((A, \alpha)\) be an effective \(\Sigma\)-algebra such that \(\Omega_\alpha\) is r.e., and let \(\{\equiv_n\}_n\) be a family of separating congruences on \(A\) such that \(\equiv_n\) is \(\alpha\)-computable uniformly in \(n\). Let \(\beta\) be a numbering of \(A_{k,\alpha}\) such that \(\phi_n\) is \((\beta, \alpha_n)\)-computable, uniformly in \(n\). Then the metric \(d\) is \((\beta^2, \rho)\)-computable.

Note that \(\alpha\) is an example of such a \(\beta\).

Proof. From the proof of Proposition 5.10(ii) we have for \(i, j \in \Omega_j\),
\[
\beta(i) \equiv_n \beta(j) \iff \alpha(\text{aprx}(i, n)) \equiv_n \alpha(\text{aprx}(j, n)).
\]

It follows that \(\equiv_n\) on \(A_{k,\alpha}\) is \(\beta\)-computable, uniformly in \(n\). Using Proposition 2.4, we define \((\beta^2, \text{id})\)-computable \(v : A^2 \to \mathbb{N}\) by
\[
v(x, y) \simeq \begin{cases} 0, & \text{if } x \neq y; \\
(\nu n.x \neq_{n+1} y \land x \equiv_n y) + 1, & \text{if } x \equiv_0 y.
\end{cases}
\]

Thus, \(v(x, y)\) computes the least \(n\) such that \(x \neq_n y\) if \(x \neq y\). Again, using Proposition 2.4, we define \((\beta^2 \times \text{id}, \rho)\)-computable \(g : A^2 \times \mathbb{N} \to \mathbb{R}_\kappa\) by
\[
g(x, y, n) \simeq \begin{cases} 2^{-n}, & \text{if } x \equiv_n y; \\
2^{-v(x, y)}, & \text{if } x \neq_n y.
\end{cases}
\]

The \(\rho\)-computable sequence \(\lambda n.g(x, y, n)\) is Cauchy and converges to \(d(x, y) \in \mathbb{R}_\kappa\). Thus, \(d(x, y) = \lim (\lambda n.g(x, y, n))\) is \((\beta^2, \rho)\)-computable. \(\square\)

An interesting example is the completion of a local ring. Let \(R\) be a commutative local Noetherian ring in the signature for rings and let \(m\) be its unique maximal ideal. Define \(\equiv_n\) on \(R\) by
\[
a \equiv_n b \iff a - b \in m^n.
\]

Then, by a theorem of Krull, \(\{\equiv_n\}_n\) is a family of separating congruences on \(R\). Assume that \((R, \alpha)\) is semicomputable, i.e., it is effective, \(\Omega_\alpha\) is r.e., and equality is \(\alpha\)-semicomputable. Then it is straightforward to see that \(\{\equiv_n\}_n\) is \(\alpha\)-semicomputable, uniformly in \(n\). However, it is shown in [12] that \(\{\equiv_n\}_n\) is in fact \(\alpha\)-computable, uniformly in \(n\), and hence the associated metric on \(R_{k,\alpha}\) is computable. This fact allowed us to construct an effective domain representation of the inverse limit of \(R\).

6 Conclusion

Equivalences between representations have been studied from the outset for numberings of countable sets and structures, often with surprising results, for example, Goncharov [13]. There are dozens of computable algebras with natural representations that can be shown to be standard or canonical [5–8]. Furthermore, stability and invariance is now commonplace.

However, when one works on uncountable topological structures problems with representations arise immediately. For example, the decimal representation of real numbers is hopeless for computability purposes; it is not obvious when two representations of the computable reals are equivalent or when a representation can be considered standard or canonical. This problem can be seen in early papers such as Mostowski [14], Robinson [15] and Moschovakis [3, 11]. Only a decade ago was a characterisation of the problem for the reals completed by Hertling [4].

Our paper extends this understanding to general algebras and, in particular, emphasises the role of algebraic operations in defining invariance: the choice of operations that are computable determine the class of representations to be compared. All approaches to computing in uncountable topological spaces and algebras meet the invariance under representations problem. This is the case for type-2 effectivity, see Weihrauch [10]. The problem of finding “canonical” representations was addressed by Schröder with the notion of admissibility [16, 17]. The notion of admissibility has also been considered in the setting of domains by Hamrin [18]. A general theory of equivalence of domain representations was begun by Blanck [19]. If one uses abstract programming models that depend only on the operations of the algebra then invariance of computability is immediate. The relationship between computability via concrete representations and via abstract programming models has been studied, for both uncountable and countable algebras, in Tucker and Zucker [20, 21].
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References