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Domain representations of spaces of compact subsets

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We present a method for constructing from a given domain representation of a space X with underlying domain D, a domain representation of a subspace of compact subsets of X where the underlying domain is the Plotkin powerdomain of D. We show that this operation is functorial over a category of domain representations with a natural choice of morphisms. We study the topological properties of the space of representable compact sets and isolate conditions under which all compact subsets of X are representable. Special attention is paid to admissible representations and representations of metric spaces.

1. Introduction

Scott domains (Scott 1970) provide denotational semantics for a wide range of programming languages and carry a natural notion of computability (Ershov 1977). Using domain representations (Blanck 2000; Stoltenberg-Hansen and Tucker 1995; Stoltenberg-Hansen and Tucker 2008), the domain-theoretic notion of computability can be extended to a large class of topological spaces, and, moreover, important classes of topological spaces can be characterised by the kind of domain representations they admit (Hamrin 2005). In addition to modelling computability, domains can be used to model non-determinism by means of powerdomains (Gierz *et al.* 2003). In this paper we extend Plotkin's powerdomain construction (Plotkin 1983), also known as the convex powerdomain, to domain representations. This amounts to implementing an effective notion of non-determinism on a large class of topological spaces.

We begin with a sketch of the construction. A *domain* in this paper will be a countably based algebraic cpo. A *domain representation* of a topological space X is a pair (D, δ) where D is a domain, D^R is a subset of D regarded as a topological space with the relativised Scott topology, and $\delta : D^R \to X$ is a quotient map. For simplicity, we will assume in this sketch that D^R is upwards closed in the domain ordering and X is Hausdorff. From such a domain representation, we construct the *powerdomain representation* $(\mathscr{P}(D), \delta_{\mathscr{P}})$ where $\mathscr{P}(D)$ is the Plotkin powerdomain. We use a result by Smyth (Smyth 1983) according to

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which $\mathscr{P}(D)$ can be modelled as the space of *lenses*, that is, non-empty compact subsets of D that are the intersection of a closed and a saturated set, with the Vietoris topology. The function $\delta_{\mathscr{P}} : \mathscr{P}(D)^R \to \mathscr{P}(X)$ is defined on $\mathscr{P}(D)^R = \{K \in \mathscr{P}(D) | K \subseteq D^R\}$ by $\delta_{\mathscr{P}}(K) := \delta[K]$. The *powerspace* $\mathscr{P}(X)$ is simply the image of $\mathscr{P}(D)^R$ under $\delta_{\mathscr{P}}$ with the quotient topology. The elements of $\mathscr{P}(X)$ are certain non-empty compact subsets of X, which we call *representable*.

Our main results can be summarised as follows:

(1) The construction of the powerdomain representation defines an endofunctor on the category of domain representations.

(2) The representable sets have good closure properties, and in many interesting cases (for example, retract representations and total continuous functionals), all non-empty compact sets are representable (modulo T_0), although this does not hold in general.

(3) For admissible domain representations, the powerspace is independent of the representation and hence defines an operation on the topological spaces admitting an admissible representation, that is, on the class of qcb_0-spaces.

(4) Many properties of spaces and representations are preserved by the powerdomain representation, for example, density, retract, Hausdorff and qcb_0.

(5) Representations of metric spaces lift to representations of the powerspace with the Hausdorff metric. This generalises previous results in Blanck (1999).

Smyth's characterisation of the Plotkin powerdomain mentioned above is crucial for our work. It allows us to render most proofs very short and in general topological terms hardly ever using domain-theoretic arguments. Applying the T_0-collapse not only to the representing domain, but also to the represented space, allows us to include non-Hausdorff spaces X smoothly in our construction.

Although we do not discuss computability aspects explicitly, it is clear from the constructions and proofs that all results hold effectively. The closure properties of representable compact sets hold effectively, and so does the lifting of metric spaces. Furthermore, the Plotkin powerdomain construction preserves effectivity, and the coincidence of the Scott and Vietoris topologies on the Plotkin powerdomain is given by computable transformations of the respective basic open sets.

The plan of the paper is as follows. In Section 2 we introduce the basic notions of domain theory and topology used in the paper. In Section 3 we look at some important results concerning operations on lenses, which will be useful at a later stage. Then, in Section 4, we present our construction of the powerdomain representation and study some basic preservation properties. The powerdomain representation gives us a natural notion of a representable compact subset of a topological space, given a domain representation of it, and, more generally, a representable lens. We study this notion in more detail in Section 5. In Section 6, we investigate what topological properties of the represented space are preserved by our construction. Finally, in Section 7, we look at the important case of a domain representation of a complete metric space.

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2. Background

2.1. *Basic domain theory*

It is well known that domains (that is, countably based algebraic cpos) are closed under Plotkin's power domain construction. Restricting further to the class of sfp-domains would give Cartesian closure, but this is not used in this paper. The restriction to countably based domains is useful for characterising the elements of the power domain. For background material on domains, see Abramsky and Jung (1994), Gierz *et al.* (2003) and Stoltenberg-Hansen *et al.* (1994).

2.2. *Domain representations*

We will begin with some basic definitions – see Blanck (2000) and Stoltenberg-Hansen and Tucker (2008) for more on domain representations.

A *domain with totality* is a pair (D, D^R), with D a domain (with the Scott topology) and D^R a subspace of the domain (with the subspace topology). A *domain representation* is a pair $(D, \delta : D^R \to X)$ such that (D, D^R) is a domain with totality, X is a topological space and δ is a quotient map. The representation (D, δ) is *dense* if D^R is topologically dense in D.

Let $(D, \delta : D^R \to X)$ and $(E, \varepsilon : E^R \to Y)$ be domain representations. A *representation morphism* (or a (δ, ε)-total map)

$$f : (D, \delta) \to (E, \varepsilon)$$

is a continuous map $f : D \to E$

$$\begin{tikzcd}
D \arrow[r, "f"] & E \\
D^R \arrow[u, hook, "\iota"] \arrow[r, "f|_{D^R}"] \arrow[d, "\delta"'] & E^R \arrow[u, hook, "\iota"'] \arrow[d, "\epsilon"] \\
X \arrow[r, "g"] & Y
\end{tikzcd}$$

such that

(i) $f[D^R] \subseteq E^R$; and
(ii) $\delta(x) = \delta(y) \Rightarrow \varepsilon(f(x)) = \varepsilon(f(y))$ for all $x, y \in D^R$.

The representation morphism induces a unique continuous function $g : X \to Y$ satisfying $g \circ \delta = \varepsilon \circ f|_{D^R}$.

A domain representation (D, δ) is *retract* if there exists a continuous $s : X \to D^R$ such that $\delta \circ s = \mathrm{id}_X$. A topological space has a dense retract representation if and only if it is a second countable T_0 space (Blanck 2000).

A domain representation (E, ε) of X is *admissible* if for every domain with dense totality (D, D^R) and continuous function $\varphi : D^R \to X$, there exists a continuous map $\hat{\varphi} : D \to E$ such that $\hat{\varphi}[D^R] \subseteq E^R$ and $\varepsilon(\hat{\varphi}(x)) = \varphi(x)$ for every $x \in D^R$ (Hamrin 2005).

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From a given admissible domain representation of X, it is easy to construct an admissible representation over a consistently complete domain. Moreover, an admissible representation can be chosen to be dense. A topological space X has a dense admissible representation if and only if it is a sequential T_0 space with a countable pseudobase (Hamrin 2005), which is again equivalent to X being a qcb_0 space (Schröder 2003).

Let (D, δ) and (E, ϵ) be domain representations of a space X. A *continuous reduction* of (D, δ) to (E, ϵ) is a continuous map $f : D \to E$ that induces the identity on X. Continuous reductions induce a preorder on domain representations. The notion of an admissible domain representation is equivalent to the domain representation being the largest amongst the dense domain representations with respect to the preorder of continuous reductions. This characterisation of admissibility also holds for other classes of representations, such as TTE (Weihrauch 2000) representations. The retract property is preserved by continuous reductions. This means that the retract property is also a notion of largeness under continuous reductions. Among domain representations, there is a close but subtle connection between retract representations and admissible representations (Blanck 2008). Retract domain representations of spaces are an important and common type of representation, and, as we will see, our power space construction preserves the retract property.

2.3. *Topology and the specialisation order*

Let $X = (X, \tau)$ be a topological space. The *specialisation order* $\leqslant$ on X is given by

$$x \leqslant y \iff (\forall U \in \tau)(x \in U \Rightarrow y \in U).$$

It is easy to see that the specialisation order is a preorder. If X is T_0, the specialisation order is antisymmetric and therefore a partial order. If X is T_1, the specialisation order is discrete, that is, $x \leqslant y \Rightarrow x = y$. For a domain, the specialisation order $\leqslant$ coincides with the domain ordering $\sqsubseteq$.

Let $A \subseteq X$. We define the *upper* and *lower set* with respect to the specialisation order by

$$\begin{aligned} \uparrow A &= \{x \in X : (\exists a \in A)(a \leqslant x)\} \\ \downarrow A &= \{x \in X : (\exists a \in A)(x \leqslant a)\}. \end{aligned}$$

Note that an open set must be an upper set and a closed set must be a lower set with respect to the specialisation order.

The *topological saturation* of A, that is, the intersection of all open neighbourhoods of A, coincides with $\uparrow A$. The *topological closure*, that is, the intersection of all closed sets containing A, is denoted by $\overline{A}$. We have $\downarrow A \subseteq \overline{A}$, but this may, in general, be a strict inclusion.

We use $\mathscr{H}(X)$ to denote the set of non-empty compact subsets of X.

A *lens* is a non-empty subset of X that can be written as the intersection of a closed set and a compact saturated set. A lens $L \subseteq X$ is itself compact and has a canonical representation of the form $L = \overline{L} \cap \uparrow L$. Let $\mathrm{Lens}(X)$ be the set of lenses in X. Clearly, $\mathrm{Lens}(X) \subseteq \mathscr{H}(X)$, and equality holds if X is a T_1 space.

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For a non-empty compact $K \subseteq X$, we define the *lens closure* of K as the set $\overline{K} \cap \uparrow K$, and use $\langle K \rangle_X$ to denote it. Clearly, lens closure is an operator $\langle \cdot \rangle_X : \mathscr{K}(X) \to \mathrm{Lens}(X)$. The lens closure $\langle K \rangle_X$ is the smallest lens containing K, and, in particular, $\langle \langle K \rangle_X \rangle_X = \langle K \rangle_X$. If X is T_1, then $\langle K \rangle_X = K$. For a non-empty finite (and hence compact) set A, we have $\langle A \rangle_X = \downarrow A \cap \uparrow A =$ the least convex set containing A.

Let A and B be non-empty subsets of X. The *Egli–Milner (pre)ordering* is defined by

$$A \sqsubseteq_{\mathrm{EM}} B \Leftrightarrow A \subseteq \downarrow B \wedge B \subseteq \uparrow A.$$

Finite sets A and B are equivalent with respect to $\sqsubseteq_{\mathrm{EM}}$ if and only if $\langle A \rangle_X = \langle B \rangle_X$.

The *topological Egli–Milner (pre)ordering* is defined by

$$A \sqsubseteq_{\mathrm{TEM}} B \Leftrightarrow A \subseteq \overline{B} \wedge B \subseteq \uparrow A.$$

Arbitrary non-empty compact sets A and B are equivalent with respect to $\sqsubseteq_{\mathrm{TEM}}$ if and only if $\langle A \rangle_X = \langle B \rangle_X$.

2.4. *Vietoris topology*

Let X be a topological space. If $U \subseteq X$ is open, let $U_\cap$ be the set of compacts $K \subseteq X$ with $K \cap U \neq \varnothing$ and let $U_\supseteq$ be the set of non-empty compacts $K \subseteq X$ with $K \subseteq U$. The *Vietoris topology* on $\mathscr{K}(X)$ is the topology generated by subbasic open sets $U_\cap$ and $U_\supseteq$ for all open $U \subseteq X$.

We consider $\mathscr{K}(X)$ as a topological space with the Vietoris topology, and $\mathrm{Lens}(X)$ as a subspace of $\mathscr{K}(X)$.

Lemma 2.1. The space $\mathrm{Lens}(X)$ is the T_0-collapse of $\mathscr{K}(X)$ via the collapsing map $\langle \cdot \rangle_X$.

Proof. If K_1 and K_2 have the same set of subbasic neighbourhoods of the form $U_\cap$, then $\overline{K_1} = \overline{K_2}$. If K_1 and K_2 have the same subbasic neighbourhoods of the form $U_\supseteq$, then $\uparrow K_1 = \uparrow K_2$. Thus, if K_1 and K_2 are indistinguishable in the Vietoris topology, $\langle K_1 \rangle_X = \langle K_2 \rangle_X$.

We have $\langle K \rangle_X \in U_\cap \iff K \in U_\cap$ since $\langle K \rangle_X \subseteq \overline{K}$. Furthermore, $\langle K \rangle_X \in U_\supseteq \iff K \in U_\supseteq$ since $\langle K \rangle_X \subseteq \uparrow K$. So K and $\langle K \rangle_X$ are indistinguishable in the Vietoris topology. Thus, if $\langle K_1 \rangle_X = \langle K_2 \rangle_X$, then K_1 and K_2 are indistinguishable since they are both indistinguishable from their common lens closure. □

Lemma 2.2. If X is a Hausdorff space, $\mathscr{K}(X) = \mathrm{Lens}(X)$ is a Hausdorff space also.

Proof. Choose non-empty compact subsets $K_1, K_2 \subseteq X$ and assume $K_2 \setminus K_1 \neq \varnothing$.

Choose $x \in K_2 \setminus K_1$. Since X satisfies the Hausdorff separation axiom, there are disjoint open neighbourhoods U and V of x and K_1, respectively. So $K_2 \in U_\cap$ and $K_1 \in V_\supseteq$, and $U_\cap$ and $V_\supseteq$ are disjoint since U and V are disjoint. □

3. More on lenses

Our interest in spaces of lenses lies in the following characterisation of the Plotkin powerdomain as a space of lenses.

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Let D be a domain and recall that we have assumed that our domains are countably based. Hence, by a result of Smyth (Smyth 1983, Theorem 3), we may identify the Plotkin powerdomain $\mathscr{P}(D)$ with the topological space Lens(D). We will use both notations, depending on context. The specialisation order on Lens(D) is $\sqsubseteq_{\mathrm{TEM}}$. The compact elements of the powerdomain are lenses generated by finite sets.

Although we will apply the results of this section to domains only, we have chosen to present the fully general situation here.

3.1. *Functoriality of* $\mathscr{K}(\cdot)$ *and* Lens$(\cdot)$

If $f : X \to Y$ is continuous, let $f_{\mathscr{K}}$ be the map that sends a non-empty compact set $K \subseteq X$ to the non-empty compact set $f[K] \subseteq Y$.

Lemma 3.1.

(i) The function $f_{\mathscr{K}} : (\mathscr{K}(X); \sqsubseteq_{\mathrm{TEM}}) \to (\mathscr{K}(Y); \sqsubseteq_{\mathrm{TEM}})$ is monotone.
(ii) In particular, for every $K \in \mathscr{K}(K)$, we have $\langle f[K]\rangle_Y = \langle f[\langle K\rangle_X]\rangle_Y$.

Proof. Part (ii) follows from Part (i) since $\langle K\rangle_X \equiv_{\mathrm{TEM}} K$.

Assume $K \sqsubseteq_{\mathrm{TEM}} K'$. Let $U \subseteq Y$ be an open set containing $f(x)$ for some $x \in K$. By assumption, $K \subseteq \overline{K'}$, so $f^{-1}[U]$ intersects the closure $\overline{K'}$ and therefore also K'. Hence, $f[K] \subseteq \overline{f[K']}$. To show that $f[K'] \subseteq {\uparrow} f[K]$, we show that every open neighbourhood of $f[K]$ contains $f[K']$. Let $U \supseteq f[K]$ be open. Then $f^{-1}[U] \supseteq K$ is open, and therefore $f^{-1}[U] \supseteq K'$. Thus, $U \supseteq f[K']$. We have shown $f[K] \sqsubseteq_{\mathrm{TEM}} f[K']$. □

Let $f : X \to Y$ be a continuous map. We define the lifting of f to the lens spaces, $f_{\mathscr{L}} : \mathrm{Lens}(X) \to \mathrm{Lens}(Y)$, by $f_{\mathscr{L}}(L) = \langle f[L]\rangle_Y$.

Lemma 3.2. $\mathrm{id}_{\mathscr{L}} = \mathrm{id}$ and $(g \circ f)_{\mathscr{L}} = g_{\mathscr{L}} \circ f_{\mathscr{L}}$, where $f : X \to Y$ and $g : Y \to Z$.

Proof. The first assertion is obvious.

For the second, let $L \in \mathrm{Lens}(X)$. By Lemma 3.1(ii), we have

$$(g \circ f)_{\mathscr{L}}(L) = \langle g[f[L]]\rangle_Z = \langle g[\langle f[L]\rangle_Y]\rangle_Z = (g_{\mathscr{L}} \circ f_{\mathscr{L}})(L).$$ □

The next two results imply that $f_{\mathscr{L}}$ is continuous.

Proposition 3.3. $f_{\mathscr{K}} : \mathscr{K}(X) \to \mathscr{K}(Y)$ is continuous.

Proof. We show that $(f_{\mathscr{K}})^{-1}$ maps Vietoris basic opens in Y to Vietoris basic opens in X. Let $U \subseteq Y$ be open. Clearly, $U \cap f[K] \neq \emptyset$ if and only if $f^{-1}[U] \cap K \neq \emptyset$, and $U \supseteq f[K]$ if and only if $f^{-1}[U] \supseteq K$. Therefore, $(f_{\mathscr{K}})^{-1}[U_{\cap}] = (f^{-1}[U])_{\cap}$ and $(f_{\mathscr{K}})^{-1}[U_{\supseteq}] = (f^{-1}[U])_{\supseteq}$. □

We consider Lens(X) as a subspace of the topological space $\mathscr{K}(X)$ with the Vietoris topology.

Proposition 3.4. $f_{\mathscr{L}} : \mathrm{Lens}(X) \to \mathrm{Lens}(Y)$ is continuous.

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Proof. Recall that $f_{\mathscr{L}} = \langle\cdot\rangle_Y \circ f_{\mathscr{H}}|_{\mathrm{Lens}(X)}$, which is a composition of continuous maps, by Lemma 2.1 and Proposition 3.3. $\square$

Specialising Proposition 3.4 to domains, we get that the Plotkin powerdomain operation defines an endofunctor over the category of domains with continuous functions as morphisms, as shown in Abramsky and Jung (1994) and Gierz *et al.* (2003).

Corollary 3.5. Let D, E be domains and $f : D \to E$ be continuous. Then $f_{\mathscr{L}} : \mathscr{P}(D) \to \mathscr{P}(E)$ is continuous.

3.2. *Lenses over a subspace*

Here we will look at the lenses over a subspace and identify these as a subset of the lenses over the full space. The motivation for this is the subset D^R of representing elements of a domain representation $(D, \delta : D^R \to X)$. However, since the results are purely topological, the exposition will use general topological spaces.

Let X be a topological space and Y be a non-empty subspace of X with the relative topology. Recall that the lens spaces $\mathrm{Lens}(X)$ and $\mathrm{Lens}(Y)$ are considered as topological spaces with the Vietoris topology.

Lemma 3.6. Let $K \subseteq Y$ be non-empty and compact.

(i) $\langle K\rangle_Y = \langle K\rangle_X \cap Y$.
(ii) $\langle K\rangle_X = \langle\langle K\rangle_Y\rangle_X$.

Proof.

(i) This is a straightforward exercise for the reader using the fact that open sets in Y are relativised open sets in X.
(ii) Clearly, $K \subseteq \langle K\rangle_Y$ and by Part (i), $\langle K\rangle_Y \subseteq \langle K\rangle_X$. Applying the monotone lens operator, we get $\langle K\rangle_X \subseteq \langle\langle K\rangle_Y\rangle_X \subseteq \langle\langle K\rangle_X\rangle_X = \langle K\rangle_X$. The statement can also be viewed as a special case of Lemma 3.1 (ii). $\square$

The inclusion function $\iota : Y \to X$ is continuous, so the map $\iota_{\mathscr{L}} : \mathrm{Lens}(Y) \to \mathrm{Lens}(X)$ is continuous by Proposition 3.4. We say that the subspace $\iota_{\mathscr{L}}[\mathrm{Lens}(Y)]$ of lenses in $\mathrm{Lens}(X)$ are the Y*-generated lenses* and we use $Y\text{-}\mathrm{Lens}(X)$ to denote them.

Lemma 3.7. Let L be a Y-generated lens, that is, $L = \langle M\rangle_X$ for some lens $M \subseteq Y$. Then $L \cap Y = M \in \mathrm{Lens}(Y)$ and $L = \langle L \cap Y\rangle_X$.

Proof. Clearly, $M \subseteq L \cap Y$. Suppose $y \in Y$ belongs to $\langle M\rangle_X$. Then y belongs to the saturation and the closure of M in X, and, *a fortiori*, y belongs to the saturation and closure of M in Y, implying that $y \in M$. Hence, $L \cap Y = M$. Thus,

$$L = \langle M\rangle_X = \langle L \cap Y\rangle_X \subseteq \langle L\rangle_X = L.$$

$\square$

Theorem 3.8. Let Y be a non-empty subspace of X. Then $Y\text{-}\mathrm{Lens}(X) \cong \mathrm{Lens}(Y)$.

Proof. Let $f : Y\text{-}\mathrm{Lens}(X) \to \mathrm{Lens}(Y)$ be defined by $f(L) = L \cap Y$. By Lemma 3.7, the function f is well defined. By Lemma 3.6, we have $f \circ \iota_{\mathscr{L}}(L) = \langle L\rangle_X \cap Y = \langle L\rangle_Y = L$ for

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all $L \in \mathrm{Lens}(Y)$, so $f \circ \imath_{\mathscr{L}} = \mathrm{id}$. By Lemma 3.7, we have $\imath_{\mathscr{L}} \circ f(L) = \langle L \cap Y \rangle_X = L$ for all $L \in Y\text{-}\mathrm{Lens}(X)$, so $\imath_{\mathscr{L}} \circ f = \mathrm{id}$.

We have already established the continuity of $\imath_{\mathscr{L}}$, so all we need to do now is show the continuity of f. Let V be an open set in Y, and let U be some open set in X such that $V = U \cap Y$. We will show that the inverse image under f of the subbasic open sets $V_\cap$ and $V_\supseteq$ are $U_\cap$ and $U_\supseteq$, respectively.

Let $L \in Y\text{-}\mathrm{Lens}(X)$. We have

$$L \in f^{-1}[V_\cap] \iff (L \cap Y) \cap V \neq \varnothing \iff L \cap V \neq \varnothing \implies L \cap U \neq \varnothing,$$

that is, $L \in U_\cap$. In the other direction, we have $L \cap U \neq \varnothing$ implies $(L \cap Y) \cap U \neq \varnothing$ since $L \subseteq \overline{L \cap Y}$. But $L \cap Y \subseteq Y$, so $(L \cap Y) \cap V \neq \varnothing$. The latter is equivalent to $L \in f^{-1}[V_\cap]$.

We also have $L \in f^{-1}[V_\supseteq] \iff L \cap Y \subseteq V$, which implies $L \subseteq U$ since $L \subseteq \uparrow(L \cap Y)$, that is $L \in U_\supseteq$. In the other direction, we have $L \subseteq U \implies L \cap Y \subseteq U \cap Y = V$, showing that $L \in f^{-1}[V_\supseteq]$. □

Lemma 3.9. Let Y be a dense subspace of X. Then $Y\text{-}\mathrm{Lens}(X)$ is a dense subspace of $\mathrm{Lens}(X)$.

Proof. Let L belong to a basic open set. The basic open sets of the Vietoris topology are finite intersections of the form $U_{1\cap} \cap \cdots U_{n\cap} \cap V_\supseteq$. Since $L \subseteq V$ and $L \cap U_i \neq \varnothing$, it must be the case that $V \cap U_i$ is a non-empty open set. Since Y is dense in X, there exists a point $y_i \in Y$ such that $y_i \in V \cap U_i$. The finite set $S = \{y_1, \ldots, y_n\}$ is a subset of V, so the Y-generated lens $L' = \langle S \rangle_X$ is also a subset of V. The point $y_i \in L'$ witnesses the fact that $L' \cap U_i \neq \varnothing$, so $L' \in U_{1\cap} \cap \cdots U_{n\cap} \cap V_\supseteq$. □

4. A powerdomain representation

Let $(D, \delta : D^R \to X)$ be a domain representation. Recall that the Plotkin powerdomain $\mathscr{P}(D)$ can be identified with $\mathrm{Lens}(D)$.

Definition 4.1. A lens $L \in \mathscr{P}(D)$ is *total* if it is D^R-generated. Let $\mathscr{P}(D)^R$ denote the subspace of total lenses, that is, the space $D^R\text{-}\mathrm{Lens}(D)$.

By Theorem 3.8, we have

$$\mathrm{Lens}(D^R) \cong \mathscr{P}(D)^R.$$

Thus, the choice of totality is essentially forced upon us. The map $\delta_{\mathscr{L}} : \mathrm{Lens}(D^R) \to \mathrm{Lens}(X)$ is a continuous map by Proposition 3.4. Define $\delta_{\mathscr{P}} : \mathscr{P}(D)^R \to \mathrm{Lens}(X)$ by composition of $\delta_{\mathscr{L}}$ with the homeomorphism above, that is, $\delta_{\mathscr{P}}(L) = \delta_{\mathscr{L}}(L \cap D^R)$. Being a composition of continuous maps, $\delta_{\mathscr{P}}$ is also continuous.

Definition 4.2. Let $(D, \delta : D^R \to X)$ be a domain representation.

(i) Let $\mathscr{P}_{(D,\delta)}(X) = \delta_{\mathscr{P}}[\mathscr{P}(D)^R]$ be the topological quotient with respect to $\delta_{\mathscr{P}}$.
(ii) Let $\mathscr{P}(D, \delta)$ denote the pair $(\mathscr{P}(D), \delta_{\mathscr{P}} : \mathscr{P}(D)^R \to \mathscr{P}_{(D,\delta)}(X))$.
(iii) A subset $A \subseteq X$ is (D, δ)-*representable*, or simply *representable*, if $A \in \mathscr{P}_{(D,\delta)}(X)$.

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Proposition 4.3. Let $(D, \delta : D^R \to X)$ be a domain representation. Then $\mathscr{P}(D, \delta)$ is a domain representation.

Proof. As $\mathscr{P}_{(D,\delta)}(X)$ is given the quotient topology, there is nothing further to show. $\square$

Note that the topology on $\mathscr{P}_{(D,\delta)}(X)$ is finer than the Vietoris topology since $\delta_{\mathscr{P}} : \mathscr{P}(D)^R \to \mathrm{Lens}(X)$ is continuous and $\mathscr{P}_{(D,\delta)}(X) \subseteq \mathrm{Lens}(X)$ is given the quotient topology.

Example 4.4. The simplest example of a domain representation is the identity map $\mathrm{id} : D \to D$ on a domain D. Then $\mathscr{P}_{(D,\mathrm{id})}(D)$ is homeomorphic to the Plotkin powerdomain over D via $\mathrm{id}_{\mathscr{P}}$.

The following proposition is the main technical step in showing the functoriality of the powerspace operator $\mathscr{P}$.

Proposition 4.5. Let $f : (D, \delta : D^R \to X) \to (E, \varepsilon : E^R \to Y)$ be a representation morphism. Then $f_{\mathscr{L}} : \mathscr{P}(D, \delta) \to \mathscr{P}(E, \varepsilon)$ is a representation morphism.

Proof. By assumption, the left-hand diagram below commutes; we will show that the right-hand diagram also commutes.

$$\begin{tikzcd}
D \arrow[r, "f"] & E \\
D^R \arrow[u, hook, "\iota"] \arrow[r, "f|_{D^R}"] \arrow[d, "\delta"'] & E^R \arrow[u, hook, "\iota"'] \arrow[d, "\epsilon"] \\
X \arrow[r, "g"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{P}(D) \arrow[r, "f_{\mathscr{L}}"] & \mathscr{P}(E) \\
\mathscr{P}(D)^R \arrow[u, hook, "\iota"] \arrow[r, "f_{\mathscr{L}}|_{\mathscr{P}(D)^R}"] \arrow[d, "\delta_{\mathscr{P}}"'] & \mathscr{P}(E)^R \arrow[u, hook, "\iota"'] \arrow[d, "\epsilon_{\mathscr{P}}"] \\
\mathscr{P}_{(D,\delta)}(X) \arrow[r, "g_{\mathscr{L}}"] & \mathscr{P}_{(E,\epsilon)}(Y)
\end{tikzcd}$$

The intention is to show that the lens lifting of the domain function should be the representation morphism. However, the lens lifting can be applied to two different but related maps, namely, f and $f|_{D^R}$. The different lens liftings need to be related to show that elements of $\mathscr{P}(D)^R$ are mapped to $\mathscr{P}(E)^R$. Let $L \in \mathscr{P}(D)^R$, that is, there exists $M \subseteq D^R$ such that $L = \langle M \rangle_D$.

$$\begin{aligned}
f_{\mathscr{L}}(L) &= f_{\mathscr{L}}(\langle M \rangle_D) \\
&= \langle f[\langle M \rangle_D] \rangle_E && \text{(by the definition of } f_{\mathscr{L}}\text{)} \\
&= \langle f[M] \rangle_E && \text{(by Lemma 3.1(ii))} \\
&= \langle f|_{D^R}[M] \rangle_E && (M \subseteq D^R) \\
&= \langle \langle f|_{D^R}[M] \rangle_{E^R} \rangle_E && \text{(by Lemma 3.6(ii))} \\
&= \langle (f|_{D^R})_{\mathscr{L}}(M) \rangle_E. && \text{(by the definition of } (f|_{D^R})_{\mathscr{L}}\text{).}
\end{aligned}$$

From either of the last two lines, it is clear that $f_{\mathscr{L}}(L) \in E^R\text{-}\,\mathrm{Lens}(E) = \mathscr{P}(E)^R$.

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Let $g : X \to Y$ be the unique continuous map such that $g \circ \delta = \varepsilon \circ f|_{D^R}$. Now let $L \in \mathscr{P}(D)^R$, that is, there exists $M \subseteq D^R$ such that $L = \langle M \rangle_D$.

$$\begin{aligned}
g_{\mathscr{L}}(\delta_{\mathscr{P}}(L)) &= g_{\mathscr{L}}(\delta_{\mathscr{L}}(M)) \\
&= (g \circ \delta)_{\mathscr{L}}(M) && \text{(by Lemma 3.2)} \\
&= \langle g \circ \delta[M] \rangle_Y \\
&= \langle \epsilon \circ f|_{D^R}[M] \rangle_Y && \text{(by assumption)} \\
&= \epsilon_{\mathscr{L}}(\langle f|_{D^R}[M] \rangle_{E^R}) \\
&= \epsilon_{\mathscr{P}}(\langle \langle f|_{D^R}[M] \rangle_{E^R} \rangle_E) \\
&= \epsilon_{\mathscr{P}}(f_{\mathscr{L}}(L)). && \text{(by the derivation above).}
\end{aligned}$$

This shows that $f_{\mathscr{L}} : \mathscr{P}(D) \to \mathscr{P}(E)$ represents $g_{\mathscr{L}} : \mathscr{P}_{(D,\delta)}(X) \to \mathscr{P}_{(E,\epsilon)}(Y)$. □

Let (D, δ) and (E, ϵ) be domain representations of X. If $f : D \to E$ is a continuous reduction and $\mathscr{P}_{(D,\delta)}(X) = \mathscr{P}_{(E,\varepsilon)}(X)$, then $f_{\mathscr{L}} : \mathscr{P}(D) \to \mathscr{P}(E)$ is a continuous reduction.

Theorem 4.6. The powerspace operator $\mathscr{P}$ is an endofunctor over the category of domain representations with representation morphisms.

Proof. By Proposition 4.5, if $f : (D, \delta) \to (E, \epsilon)$ is a representation morphism, $\mathscr{P}(f) = f_{\mathscr{L}}$ is a representation morphism from $\mathscr{P}(D, \delta)$ to $\mathscr{P}(E, \epsilon)$.

Functoriality holds by Lemma 3.2. □

5. Representable subsets

Let $(D, \delta : D^R \to X)$ be a domain representation. We study which subsets of X are representable.

The following lemma shows that nothing further would be representable even if we relaxed our totality to arbitrary non-empty compact subsets of D^R instead of D^R-generated lenses.

Lemma 5.1. If $A = \langle \delta[K] \rangle_X$ for some non-empty compact set $K \subseteq D^R$, then A is representable.

Proof. The lens $\langle K \rangle_D$ is D^R-generated. Thus,

$$\delta_{\mathscr{P}}(\langle K \rangle_D) = \delta_{\mathscr{L}}(\langle K \rangle_D \cap D^R) = \delta_{\mathscr{L}}(\langle K \rangle_{D^R}) = \langle \delta[\langle K \rangle_{D^R}] \rangle_X = \langle \delta[K] \rangle_X = A,$$

where the penultimate equation holds by Lemma 3.1 (ii). □

Lemma 5.2. If $A_1, \ldots, A_n$ are representable, then $\langle \bigcup_{i=1}^n A_i \rangle_X$ is representable.

Proof. Assume that $L_i \in \mathscr{P}(D)^R$ satisfies $\delta_{\mathscr{P}}(L_i) = A_i$ for $1 \leqslant i \leqslant n$. Let $A = \bigcup_{i=1}^n A_i$ and $L = \langle \bigcup_{i=1}^n L_i \rangle_D$. Then $\delta[\uparrow L \cap D^R] \subseteq \uparrow A$ by the monotonicity of δ with respect to the specialisation order, and $\delta[\overline{L} \cap D^R] \subseteq \overline{A}$ by the continuity of δ. Hence, $A \subseteq \delta[L \cap D^R] \subseteq \langle A \rangle_X$, that is, $\langle A \rangle_X = \langle \delta[L \cap D^R] \rangle_X = \delta_{\mathscr{P}}(L)$. □

Lemma 5.3. Any finitely generated lens $A \subseteq X$ is representable.

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Proof. Any singleton set $\{x\}$ is representable since we may choose any $d \in \delta^{-1}(x)$, and, clearly, $\delta_{\mathscr{P}}(\{d\}) = \{x\}$. The result now follows from Lemma 5.2. $\square$

Example 5.4. Let X be a discrete space, so $\mathrm{Lens}(X) = \wp^*_{\mathrm{f}}(X)$, the set of finite non-empty subsets of X. By Lemma 5.3, every finite non-empty set is representable. Hence, $\mathscr{P}_{(D,\delta)}(X) = \wp^*_{\mathrm{f}}(X) = \mathrm{Lens}(X)$.

Lemma 5.5. Non-empty relatively closed subsets of representable sets are representable.

Proof. Let $L \in \mathscr{P}(D)^R$ and $A = \delta_{\mathscr{P}}(L)$. Let C be a closed set intersecting A. By continuity, $\delta^{-1}[C]$ is closed in D^R. The lens $L' = \langle L \cap \delta^{-1}[C]\rangle_D$ is D^R-generated and $\delta_{\mathscr{P}}(L') = A \cap C$. $\square$

We can show that for the class of retract representations, all lenses are representable.

Lemma 5.6. Let $(D, \delta : D^R \to X)$ be a retract representation. Then $\mathscr{P}_{(D,\delta)}(X) = \mathrm{Lens}(X)$, and the topologies coincide.

Proof. Let $s : X \to D^R$ be a continuous map such that $\delta \circ s = \mathrm{id}_X$, and let $A \subseteq X$ be a lens. Then

$$\delta_{\mathscr{L}}(s_{\mathscr{L}}(A)) = \delta_{\mathscr{L}}(\langle s[A]\rangle_D) = \langle \delta[s[A]]\rangle_X = \langle A\rangle_X = A.$$

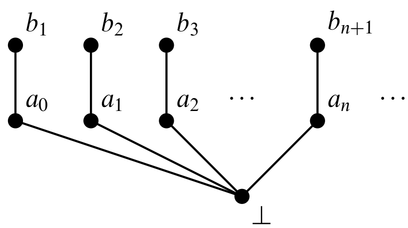
Since $\delta_{\mathscr{L}} \circ s_{\mathscr{L}} : \mathrm{Lens}(X) \to \mathscr{P}_{(D,\delta)}(X)$ is continuous, it follows that the topology on $\mathrm{Lens}(X)$ is finer than the topology on $\mathscr{P}(X)$. But, since the other direction is known in general, the two topologies must coincide. $\square$

According to Escardó *et al.* (2004), a compact *qcb* space X with the Hausdorff property is countably based. In this case, there exists a dense retract representation $(D, \delta : D^R \to X)$ (Blanck 2000), and, therefore, by Lemma 5.6, all compact subsets of X are (D,δ)-representable and the induced topology coincides with the Vietoris topology.

The following example shows that there are domain representations for which not all lenses are representable. It also shows that representability depends on the underlying domain representation.

Example 5.7. Let X be the natural numbers $\mathbb{N}$ with the order topology (Alexandroff topology) given by the usual ordering on $\mathbb{N}$. The open sets in this topology are the infinite intervals $[n, \infty)$, for $n \in \mathbb{N}$. The lenses over X are finite or infinite intervals of natural numbers. Note that X is T_0, but neither T_1 nor sober; it is also not a cpo.

We construct a domain representation of X. Let D be the domain



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Let $D^R = D \setminus \{\perp\}$, and define the representation map $\delta : D^R \to X$ by

$$\delta(a_i) = \delta(b_i) = i.$$

It is easy to see that δ is continuous. We need to verify that it is a quotient map. Let $A \subseteq X$ be such that its pre-image $U = \delta^{-1}[A]$ is open in D^R. If $n \in A$, then $a_n \in U$, and since U is open, we have $b_{n+1} \in U$, and thus, $n+1 \in A$. So A must be an interval $[n, \infty)$ for some n, showing that δ is a quotient map.

Clearly, there cannot exist a continuous section $s : X \to D^R$, so (D, δ) is not a retract domain representation.

Let $K \subseteq D^R$ be compact. Then K must be finite. The lens closure $\langle K \rangle_D$ is also finite. Thus, all total lenses in $\mathscr{P}(D, \delta)$ are finite. The image of $\langle K \rangle_D$ under $\delta_{\mathscr{P}}$ will be a finite interval. That is, the space $\mathscr{P}_{(D,\delta)}(X)$ only consists of finite intervals.

As a contrast, we will now construct a retract domain representation such that all lenses over X are representable. Let E be the ideal completion of $(\mathbb{N}, \leqslant)$. Let E^R be the set of compact elements. Let the representing map $\epsilon : E^R \to X$ be the identity. This is a retract domain representation of X, where the section is again the identity. By Lemma 5.6, $\mathscr{P}_{(E,\epsilon)}(X) \cong \mathrm{Lens}(X)$.

These two domain representations of X show that the powerspace depends on the chosen domain representation, and that for some domain representations there may exist lenses that are not representable.

Given an admissible representation of X, we do not know whether all lenses in X are representable, but we can say something about the lenses generated by continuous images of the Cantor space $\mathbf{C}$.

Lemma 5.8. Let $(D, \delta : D^R \to X)$ be an admissible representation. If $f : \mathbf{C} \to X$ is a continuous function, then $\langle f[\mathbf{C}] \rangle_X$ is (D, δ)-representable.

Proof. We represent $\mathbf{C}$ as a dense totality in the Cantor domain. Since (D, δ) is admissible, the continuous map $f : \mathbf{C} \to X$ lifts to a continuous $\hat{f}$ from the Cantor domain into D. Then $\hat{f}[\mathbf{C}] \subseteq D^R$ is a non-empty compact set and

$$\delta_{\mathscr{P}}(\langle \hat{f}[\mathbf{C}] \rangle_D) = \delta_{\mathscr{L}}(\langle \hat{f}[\mathbf{C}] \rangle_{D^R}) = \langle \delta[\langle \hat{f}[\mathbf{C}] \rangle_{D^R}] \rangle_X = \langle \delta[\hat{f}[\mathbf{C}]] \rangle_X = \langle f[\mathbf{C}] \rangle_X.$$

The penultimate equation holds by Lemma 3.1(ii). □

Example 5.9. Important instances of this Lemma are the domain representations of the total continuous functionals of finite types over the natural numbers. More precisely, let D_ι be the flat domain $\mathbb{N}_\perp$ and $D_{\rho\to\sigma} = [D_\rho \to D_\sigma]$, the domain of continuous functions from D_ρ to D_σ. Let D^R_ρ be the set of total elements in D_ρ, that is, $D^R_\iota = \mathbb{N}$ and $D^R_{\rho\to\sigma} = \{f \mid f[D^R_\rho] \subseteq D^R_\sigma\}$. Two total functionals f, g of the same type are equivalent if they map equivalent arguments to equivalent results. Ershov showed that the quotients of these equivalence relations define the total continuous functionals introduced by Kleene and Kreisel (Kleene 1959; Kreisel 1959; Ershov 1977). Let $\delta_\rho : D^R_\rho \to K_\rho$ denote these quotients. The domain representations (D_ρ, δ_ρ) are admissible, see, for example, Hamrin (2005, Theorem 7.6). Furthermore, Normann showed that every compact subset of K_ρ is the continuous image of a compact subspace of Baire space, and hence a continuous image

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of Cantor space (Normann 1980, Theorem 3.45 (iv)). It follows, by Lemma 5.8, that all non-empty compact subsets of K_ρ are representable. A more direct proof of this fact was also given implicitly by Escardó (2008, Lemma 3.3.4.3).

An interesting subclass of the qcb_0 spaces are the sequential Hausdorff spaces, which admit a countable pseudobase consisting of closed sets (Normann 2008). Every such space X has a dense admissible and upwards-closed domain representation $(D, \delta : D^R \to X)$, where upwards-closed means that if $x \in D^R$ and $x \sqsubseteq x'$, then $x' \in D^R$ and $\delta(x) = \delta(x')$. Dag Normann has proved (private communication) that in this case, every non-empty compact subset K of X can be represented as a continuous image of Cantor space and is (D, δ)-representable.

6. Topological properties

In this section we consider what, primarily topological, properties of a domain representable space are preserved by the powerspace functor. We will assume throughout that $(D, \delta : D^R \to X)$ is a domain representation of X.

Lemma 6.1. If (D, δ) is a dense domain representation, then $(\mathscr{P}(D), \delta_{\mathscr{P}})$ is a dense domain representation.

Proof. The result follows from Lemma 3.9. □

Proposition 6.2. If X is T_0, then $\mathscr{P}_{(D,\delta)}(X)$ is T_0 and therefore a qcb_0-space.

Proof. The Vietoris topology on Lens(X) is T_0, and thus so is any finer topology on the subspace $\mathscr{P}_{(D,\delta)}(X)$.

By construction, $\mathscr{P}_{(D,\delta)}(X)$ is a topological quotient of $\mathscr{P}(D)^R$, which is a countably based space. □

A qcb_0 space X is characterised by the existence of a dense admissible representation $(D, \delta : D^R \to X)$. We now prove that the powerspace $\mathscr{P}_{(D,\delta)}(X)$ is independent of the specific choice of dense admissible representation.

Lemma 6.3. Let $(D, \delta : D^R \to X)$ and $(E, \varepsilon : E^R \to X)$ be dense admissible representations of X.

Then $\mathscr{P}_{(D,\delta)}(X) = \mathscr{P}_{(E,\varepsilon)}(X)$. Moreover, the powerspace representations $\mathscr{P}(D, \delta)$ and $\mathscr{P}(E, \varepsilon)$ reduce to each other.

Proof. Since (D, δ) is dense and (E, ε) is admissible, there exists a continuous reduction $f : (D, \delta) \to (E, \varepsilon)$. The representation morphism $f_{\mathscr{L}} : \mathscr{P}(D, \delta) \to \mathscr{P}(E, \varepsilon)$ represents the continuous inclusion map $\mathrm{id}_{\mathscr{L}} : \mathscr{P}_{(D,\delta)}(X) \subseteq \mathscr{P}_{(E,\varepsilon)}(X)$. By symmetry, the reverse inclusion map is continuous as well, and $\mathscr{P}_{(D,\delta)}(X) = \mathscr{P}_{(E,\varepsilon)}(X)$.

This means that $f_{\mathscr{L}} : \mathscr{P}(D, \delta) \to \mathscr{P}(E, \varepsilon)$ represents the identity map, so it is a continuous reduction. Symmetrically, we also have a continuous reduction of $\mathscr{P}(E, \varepsilon)$ to $\mathscr{P}(D, \delta)$. □

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We use $\mathscr{P}(X)$ to denote this representation independent powerspace of X. Note that it cannot be considered as a subspace of Lens(X), as the topology is, in general, finer than the subspace topology.

By Proposition 6.2, the class of qcb_0 spaces is closed under the powerspace operation $\mathscr{P}$. We now show that $\mathscr{P}$ is an endofunctor over the category of qcb_0 spaces and continuous functions.

Lemma 6.4. Let X, Y be qcb_0 spaces and $f : X \to Y$ be continuous. Then $f_{\mathscr{L}} : \mathscr{P}(X) \to \mathscr{P}(Y)$ is continuous.

Proof. Choose dense admissible representations $(D, \delta : D^R \to X)$ and $(E, \varepsilon : E^R \to Y)$. Let $\varphi = f \circ \delta : D^R \to Y$, which is a continuous map.

Since (D, D^R) is dense and (E, ε) is admissible, there exists a continuous $\hat{\varphi} : D \to E$ satisfying $\hat{\varphi}[D^R] \subseteq E^R$ and $\varepsilon \circ \hat{\varphi}|_{D^R} = \varphi$. In particular, $\hat{\varphi} : (D, \delta) \to (E, \varepsilon)$ is a representation morphism of f.

$$\begin{tikzcd}
D \arrow[r, dashed, "\hat{\varphi}"] & E \\
D^R \arrow[u, hook, "\iota"] \arrow[r, dashed, "\hat{\varphi}|_{D^R}"] \arrow[d, "\delta"'] \arrow[dr, "\varphi"] & E^R \arrow[u, hook, "\iota"'] \arrow[d, "\epsilon"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

The representation morphism $\hat{\varphi}_{\mathscr{L}} : \mathscr{P}(D, \delta) \to \mathscr{P}(E, \varepsilon)$ induces a unique continuous map $g : \mathscr{P}(X) \to \mathscr{P}(Y)$ satisfying $g \circ \delta_{\mathscr{P}} = \varepsilon_{\mathscr{P}} \circ \hat{\varphi}_{\mathscr{L}}|_{\mathscr{P}(D)^R}$.

On the other hand, if $L \in \mathscr{P}(D)^R$, then

$$\begin{aligned} f_{\mathscr{L}}(\delta_{\mathscr{P}}(L)) &= \varphi_{\mathscr{L}}(L \cap D^R) = (\varepsilon \circ \hat{\varphi}|_{D^R})_{\mathscr{L}}(L \cap D^R) \\ &= \varepsilon_{\mathscr{P}}(\hat{\varphi}_{\mathscr{L}}(L)), \end{aligned}$$

which shows that $f_{\mathscr{L}} \circ \delta_{\mathscr{P}} \varepsilon_{\mathscr{P}} \circ \hat{\varphi}_{\mathscr{L}}|_{\mathscr{P}(D)^R}$. Since $\delta_{\mathscr{P}}$ is a representation map, this implies $g = f_{\mathscr{L}}$, showing that $f_{\mathscr{L}}$ is continuous. $\square$

It is an open problem whether $\mathscr{P}(D, \delta)$ is admissible when $(D, \delta : D^R \to X)$ is. Every other domain representation of $\mathscr{P}(X)$ of type $\mathscr{P}(E, \varepsilon)$, where (E, ε) is some dense domain representation, will be continuously reducible to $\mathscr{P}(D, \delta)$. It is, however, conceivable that there are dense domain representations of $\mathscr{P}(X)$ that are not, and in that case, $\mathscr{P}(D, \delta)$ need not be admissible. Clearly, there is some dense admissible domain representation of the qcb_0 space $\mathscr{P}(X)$, but not necessarily over a powerdomain $\mathscr{P}(D)$.

A strictly smaller class of topological spaces are the second countable T_0 spaces, which are characterised by the existence of a dense retract representation. For retract representations, we do obtain the preservation property that we lack for admissible representations.

Proposition 6.5. Let $(D, \delta : D^R \to X)$ be a retract domain representation of X. Then $\mathscr{P}(D, \delta)$ is a retract domain representation of Lens(X).

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Proof. Let $s : X \to D^R$ be a continuous maps satisfying $\delta \circ s = \mathrm{id}_X$. From Proposition 3.4, we know that $s_{\mathscr{L}} : \mathrm{Lens}(X) \to \mathrm{Lens}(D^R)$ is continuous. Moreover, $\mathscr{P}(D)^R$ is homeomorphic to $\mathrm{Lens}(D^R)$, so it is sufficient to show that $\delta_{\mathscr{L}} \circ s_{\mathscr{L}} = \mathrm{id}_{\mathrm{Lens}(X)}$, which follows from the proof of Lemma 5.6. □

In combination with Lemma 6.1, this shows that second countable T_0 spaces are closed under $\mathscr{P}$. Moreover, the powerspace $\mathscr{P}(X)$ is defined independently of the particular choice of retract (D, δ), regardless of density.

Finally, we observe that the Hausdorff separation axiom is preserved by our powerspace construction.

Proposition 6.6. If X is a Hausdorff space, $\mathscr{P}_{(D,\delta)}(X)$ is Hausdorff also.

Proof. The topology on $\mathscr{P}_{(D,\delta)}(X)$ is finer than the subspace topology from $\mathscr{H}(X)$, which is Hausdorff by Lemma 2.2. □

7. The powerspace of a metric space

Let (X, d) be a metric space. The *Hausdorff distance* of two non-empty compact subsets K, K' is defined by

$$d_H(K, K') = \max\{\sup_{x \in K} d(x, K'), \sup_{x \in K'} d(x, K)\}$$

where $d(x, K) = \inf_{y \in K} d(x, y)$. The following lemma is well known and easy to prove.

Lemma 7.1. The Hausdorff distance defines a metric on $\mathscr{H}(X)$. Its topology coincides with the Vietoris topology.

Let $D_{\mathbb{R}}$ be the interval domain, that is, the ideal completion of the closed rational intervals in $\mathbb{Q} \cup \{+\infty, -\infty\}$ ordered by reverse inclusion. Let $D_{\mathbb{R}}^R$ be the set of ideals whose intersection is a singleton, and let $\delta_{\mathbb{R}} : D_{\mathbb{R}}^R \to \mathbb{R}$ be the map that selects the unique element. Then $(D_{\mathbb{R}}, \delta_{\mathbb{R}})$ is a dense retract, and hence admissible, representation of the reals. In the following, any admissible domain representation of the reals could be taken instead, but the interval representation is particularly convenient to work with.

Blanck (1999) showed that it is possible to build a domain representation of $\mathscr{H}(X)$ by taking the powerdomain of the standard domain representation of X constructed in Blanck (1997). We aim to generalise this result by applying our powerfunctor to general domain representations of metric spaces. However, for this to go through we need the domain representation of the metric space to be considered as a topological algebra, and not just as a topological space. Hence, we will also require that the metric is representable.

Definition 7.2. A *domain representation* of a metric space (X, d) is a pair

$$((D, \delta : D^R \to X), \bar{d} : D^2 \to D_{\mathbb{R}}),$$

where (D, δ) is a retract domain representation of X, and $\bar{d}$ satisfies $\bar{d}[(D^R)^2] \subseteq D_{\mathbb{R}}^R$, and

$$d(\delta(x), \delta(x')) = \delta_{\mathbb{R}} \bar{d}(x, x')$$

for all $x, x' \in D^R$.

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The restriction to retract domain representations is justified since it follows from the results and constructions in Blanck (1997) that all separable metric spaces have retract domain representations in the sense above.

In the following, let $((D, \delta), \bar{d} : D^2 \to D_{\mathbb{R}})$ be a domain representation of a metric space (X, d).

Consider the powerfunctor on a domain representation of a metric space. That is, the domain representation $\mathscr{P}(D, \delta)$. We aim to show that this representation can be extended to a domain representation of the space $\mathscr{H}(X)$ with the Hausdorff metric. From Proposition 6.5 and Lemma 7.1, the space $\mathscr{H}(X)$ is the same as the space $\mathscr{P}(X)$ and carries the topology induced by the Hausdorff metric. Hence, all that is left to show is that the Hausdorff metric can be tracked by a continuous function $\bar{d}_H : (\mathscr{P}D)^2 \to D_{\mathbb{R}}$ such that $\bar{d}_H[(\mathscr{P}(D)^R)^2] \subseteq D^R_{\mathbb{R}}$, and $d_H(\delta_{\mathscr{P}}(K), \delta_{\mathscr{P}}(K')) = \delta_{\mathbb{R}} \bar{d}_H(K, K')$.

We will now carry out the construction of $\bar{d}_H$.

Lemma 7.3. Let x, y be elements of D^R. For any $\varepsilon > 0$, there exist $a \in \mathrm{approx}(x)$ and $b \in \mathrm{approx}(y)$ such that $\bar{d}(a, b)$ is an interval of length less than ε containing $d(\delta(x), \delta(y))$.

Proof. Let (r, s) be an open interval containing $\delta\bar{d}(x, y) = d(\delta(x), \delta(y))$. The set U of subintervals of (r, s) is open in the interval domain. The set $\bar{d}^{-1}[U]$ is an open subset of D^2 containing (x, y). Thus, there exist compact a and b in $\bar{d}^{-1}[U]$ below x and y, respectively. □

Consider the powerfunctor on a domain representation of a metric space. That is, the domain representation $(\mathscr{P}(D), \delta_{\mathscr{P}} : \mathscr{P}(D)^R \to \mathscr{P}(X))$. We aim to show that the Hausdorff metric is a continuous function from $\mathscr{P}(X)^2$ to $D_{\mathbb{R}}$.

The compact elements of $\mathscr{P}(D)$ can be identified as the equivalence classes of finite sets of compact elements in D. A canonical choice in each equivalence class is the convex closure of the finite set, but this set need not be finite. We will tacitly assume that A and B range over $\wp^*_{\mathrm{f}}(D_c)$, the set of finite non-empty subsets of D_c. We will also tacitly assume that a and b range over D_c.

We will need to take minimums and maximums over non-empty sets of intervals. These are defined as

$$\min\{[s_i, t_i] : i \in I\} = [\min\{s_i : i \in I\}, \min\{t_i : i \in I\}],$$

and analogously for max. Note that if the operators are viewed as n-ary operators on the interval domain, for $n \geqslant 1$, then they are monotone in each argument.

It is common, and often useful, to define distances in metric spaces between objects other than points in the space. It is also customary to abuse the notation and retain the letter d for distance functions derived from the metric d. We will follow this tradition.

Define the distance $d : X \times \mathscr{P}(X) \to \mathbb{R}$ by

$$d(x, K) = \inf_{y \in K} d(x, y).$$

We mimic this definition by defining $\bar{d} : D_c \times \mathscr{P}(D)_c \to D_{\mathbb{R}}$ by

$$\bar{d}(a, B) = \min_{b \in B} \bar{d}(a, b).$$

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Assume that $a \sqsubseteq a'$. Then the minimum taken in $\bar{d}(a', B)$ is over smaller intervals, as the original $\bar{d}$ is monotone. Since the minimum operation is monotone, the resulting interval is smaller, that is, $\bar{d}$ is monotone in the first argument.

To see that $\bar{d}$ is monotone in its second argument, we need to consider the upper and lower bounds on the intervals separately. Assume that $B \sqsubseteq_{\mathrm{EM}} B'$. Let $S = \{\bar{d}(a,b) : b \in B\}$ and $S' = \{\bar{d}(a,b') : b' \in B'\}$. For each $b \in B$ there exists $b' \in B'$ such that $b \sqsubseteq b'$. Since $\bar{d}(a,b) \sqsubseteq \bar{d}(a,b')$, we have that any upper bound of an interval in S will have a better (smaller) upper bound in S', and hence $\min S'$ will have a better upper bound than $\min S$. For the lower bounds, we have that for each $b' \in B'$ there exists $b \in B$ such that $b \sqsubseteq b'$. Again, since $\bar{d}(a,b) \sqsubseteq \bar{d}(a,b')$, we have that any lower bound of an interval in S' will have a worse (smaller) lower bound in S, and hence $\min S'$ will have a better (larger) lower bound than $\min S$. Thus, $\bar{d}$ is monotone in its second argument as well. Note that the above argument actually required both directions of the Egli–Milner ordering.

The monotonicity of $\bar{d}$ with respect to the Egli–Milner ordering is enough to show that $\bar{d}$ is well defined on $\mathscr{P}(D)_c$. Moreover, it allows us to extend it to a continuous function $\bar{d} : D \times \mathscr{P}(D) \to D_{\mathbb{R}}$.

Consider an $x \in D$ and a non-empty compact $K \subseteq D^R$. We will now show that $d(\delta(x), \delta_{\mathscr{P}}(K))$ is represented by $\bar{d}(x, K)$. The value

$$d(\delta(x), \delta_{\mathscr{P}}(K)) = \inf_{y \in K} d(\delta(x), \delta(y))$$

will, in fact, be obtained for some choice of $y_0 \in K$ since K is compact.

Lemma 7.4. Let $x \in D^R$ and $K \subseteq D^R$ be compact. For any $\varepsilon > 0$ there exist $a \sqsubseteq x$ and $B \sqsubseteq_{\mathrm{EM}} K$ such that, for all $b \in B$, $\bar{d}(a,b)$ is an interval of length less than ε.

Proof. For any $y \in K$, by Lemma 7.3, there exists $a_y \sqsubseteq x$ and $b_y \sqsubseteq y$ such that $\bar{d}(a_y, b_y)$ has length less than ε. Clearly, $\{\uparrow b_y : y \in K\}$ is an open covering of K, so there exists a finite subcovering

$$\{\uparrow b_{y_1}, \ldots, \uparrow b_{y_n}\}.$$

Let

$$a = \bigsqcup \{a_{y_1}, \ldots, a_{y_n}\}$$
$$B = \{b_{y_1}, \ldots, b_{y_n}\}.$$

We have $B \sqsubseteq_{\mathrm{EM}} K$, and by the monotonicity of $\bar{d}$, we have $\bar{d}(a, b_{y_i})$ has length less than ε. $\square$

For an $\varepsilon > 0$, let a and B be as in the above lemma. Let $b \in B$ be such that $y_0 \in \uparrow b$, where $y_0 \in K$ satisfies $d(\delta(x), \delta(y_0)) = \inf_{y \in K} d(\delta(x), \delta(y))$. Then $\bar{d}(a,b)$ is an interval of length less than ε containing $d(\delta(x), \delta(y_0))$ since $\bar{d}$ represents d. The intervals $\bar{d}(a,b')$ for all other $b' \in B$ will have upper bounds greater than or equal to $d(\delta(x), \delta(y_0))$ and will have length less than ε. The minimum of all these intervals will be an interval of length less than ε approximating the distance. Since ε was arbitrary, $\bar{d}(x, K)$ will be a total element in $D_{\mathbb{R}}$. We have now shown the following result.

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Proposition 7.5. The function $\bar{d} : D \times \mathscr{P}(D) \to D_{\mathbb{R}}$ induces the function $d : X \times \mathscr{P}(X) \to \mathbb{R}$.

Now consider the distance function $d : \mathscr{P}(X)^2 \to \mathbb{R}$ defined by

$$d(K, K') = \sup_{x \in K} d(x, K'),$$

so $d_H(K, K') = \max\{d(K, K'), d(K', K)\}$. Note that, in general, $d(K, K') \neq d(K', K)$. Define $\bar{d} : \mathscr{P}(D)_c^2 \to D_{\mathbb{R}}$ by

$$\bar{d}(A, B) = \max_{a \in A} \bar{d}(a, B).$$

Clearly, $\bar{d}$ is monotone in its second argument since the point-to-set version is monotone in its second argument. That $\bar{d}$ is also monotone in its first argument follows from an argument similar to the monotonicity of the second argument of the point-to-set version of $\bar{d}$. Thus, $\bar{d}$ is well defined and may be extended to a continuous function $\bar{d} : \mathscr{P}(D)^2 \to D_{\mathbb{R}}$.

We may repeat the development above to prove the following result.

Proposition 7.6. The function $\bar{d} : \mathscr{P}(D)^2 \to D_{\mathbb{R}}$ induces the function $d : \mathscr{P}(X)^2 \to \mathbb{R}$.

Now we mimic the Hausdorff metric by defining $\bar{d}_H : \mathscr{P}(D)_c^2 \to D_{\mathbb{R}}$ by

$$\bar{d}_H(A, B) = \max\{\bar{d}(A, B), \bar{d}(B, A)\}.$$

This is clearly monotone and therefore well defined and extendable to a continuous function.

Proposition 7.7. $\bar{d}_H : \mathscr{P}(D)^2 \to D_{\mathbb{R}}$ represents $d_H : \mathscr{P}(X)^2 \to \mathbb{R}$.

Proof. The result is immediate. □

Theorem 7.8. Let $((D, \delta), \bar{d} : D^2 \to D_{\mathbb{R}})$ be a retract domain representation of a metric space (X, d). Then $(\mathscr{P}(D, \delta), \bar{d}_H : \mathscr{P}(D)^2 \to D_{\mathbb{R}})$ is a retract domain representation of the metric space $(\mathscr{H}(X), d_H)$.

Proof. The result follows from Lemma 7.1 and Propositions 7.7 and 6.5. □

8. Conclusions and further work

The main motivation for the powerspace construction presented here is its natural characterisation as a space of certain compact subsets. In the important example of a separable metric space, it gives the expected result, that is, the set of non-empty compact subsets with the Hausdorff metric, thus generalising the result of Blanck (1999).

A countable domain is a domain representation of itself with the identity as the representation map. Our powerspace construction will in this case yield the Plotkin powerdomain. Thus, our construction is conservative with respect to the usual powerdomains.

The powerdomain is the space of lenses with the Vietoris topology (Smyth 1983). As we have seen, our construction will give a space of lenses with the Vietoris topology in many cases. It is not clear whether this is always true in general. That is, there may exist topological spaces for which our powerspace will have a topology that is strictly finer

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than the Vietoris topology. Even if this turns out to be the case, there is still scope to provide further classes of spaces for which the induced topology is the Vietoris topology: possible candidates here would be cpos or sober spaces.

The notion of an admissible representation plays an important role since the existence of such a representation ensures that the powerspace construction is functorial (with respect to continuous functions). However, we do not know that the powerspace of an admissible domain representation will again be admissible. This is the main open problem left by this paper. For the easier situation of retract representations, we know that the powerspace will again be a retract representation.

There is an alternative characterisation of the Plotkin powerdomain as the free algebra with respect to the Plotkin powertheory (see Abramsky and Jung (1994) and Gierz *et al.* (2003) for details), and a similar free algebra construction has been studied for qcb_0 spaces (Battenfeld 2006). A clear advantage of such an algebraic definition is that it points out the necessity of considering the space constructed for representing non-determinism. An interesting project would be to compare this construction to ours since they both preserve the property of being qcb_0. We claim that the connection with the topological characterisation of the Plotkin powerdomain and the preservation of the Hausdorff separation axiom are weighty arguments in favour of our choice.

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